Construction of a Topological Group Based on Elliptic Functions

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Abstract
The aim of this paper is the construction of a topological group; for this purpose we have taken a space whose elements are elliptic functions (doubly periodic meromorphic functions). And we have focused on the set of all elliptic functions of the same periods.

Keywords: Elliptic Function, Topological Space, Topological Group.

Introduction:

A topological group is a topological space X, which is also a group, in which the group multiplication and the taking of inverses, are continuose[1,2]. More precisely, the maps $\omega : X \times X \to X$ defined by $\omega(g,h) = gh$, $i : X \to X$ defined by $i(g) = g^{-1}$; $g,h \in X$, are continuous[1,2]. Sahleh and Sanatee in 2008, show if the topological fundamental group of X, is compact then it can be embedded in the inverse limit of discrete groups[3]. Also Dehkordy and Malek Mohamad in 2010 presented a generalized necessary and sufficient condition for a topological fundamental group to be discrete [4]. In this work we concentrated on the set E (the set of all elliptic functions of the same periods, on the field of complex numbers.), which is defined in [5], that contains an analyticaly study of this set. A function $f : C \to C_\infty$ with two periods $2w_1$ and $2w_3$, the ratio $\frac{2w_1}{2w_3}$ of which is not real, is called doubly periodic[6]. A function that is analytic in a region D, except for poles in D, is called meromorphic in D[7,8]. A doubly periodic meromorphic function is called elliptic. A function which is analytic every where on the whole plane C is called entire[7,8].

<u>Proposition 1:</u> Let $f(z)$ and $g(z)$ be two periodic functions on C (the set of all complex numbers) of fundamental periods $a = \alpha + i\beta$ and $b = \alpha' + i\beta'$ respectively, where α, α', β and β' are real numbers and $a \neq b$. Then we prove some cases that f and g have the common periods.

<u>Proof:</u> Case 1: If $\begin{cases} \alpha = \beta \\ \alpha' = \beta' \end{cases}$; α, α', β and β' are integers and $g.c.d.(\alpha, \alpha') = 1$, then the common period is $d = \alpha' a = \alpha b$.

Case 2: If $\begin{cases} \alpha = \beta \\ \alpha' = \beta' \end{cases}$; α, α', β and β' are integers and $g.c.d(\alpha, \alpha') = c \neq 1$, then the common period is $d = \alpha' a / c = \alpha b / c$.

Case 3: If $\begin{cases} \alpha \neq \beta \\ \alpha' \neq \beta' \end{cases}$ and

a. If $a = kb$, where k is integer then $d = a$.

b. If $b = k'a$, where k' is integer then $d = b$.

Case 4: If $\begin{cases} \alpha \neq \beta \\ \alpha' \neq \beta' \end{cases}$ an

a. If $a = kb$, where $k = r / s$; r and s are integers and $g.c.d(r,s) = 1$ then $d = sa = rb$.

$f_i, f_j \in F_1; i \neq j; i, j = 1,2,\cdots\}$, b. If $b = k'a$, where $k' = r'/s'$; r' and s' are integers and $g.c.d(r', s') = 1$ then $d = s'b = r'a$

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Case 5:
If both α and α' are zero and both β and β' are rational numbers then we find the common period for f and g depending on the theorem (the addition and multiplication of two periodic functions with rational period will also be periodic). Again we depend on this theorem if β and β' are zero and α and α' are rational numbers . ◊

Remark 2:
It is obvious that the set E is closed under the operations: addition, subtruction, multiplication, division by non zero divizor, and differentiation.

To satisfy the objective of this paper, we costruct a set F and take two subsets of it, then a suitable topology on F, depending on these subsets. Let F be the set of all elliptic functions in which for every $f, g \in F$ either $f, g \in E$ or there exists a relation between their periods as follows:
For simplification we denote the periods of f and g by $2w_1, 2w'_3$ and $2w'_1, 2w_3$ respectively. Either $2w_1 = 2w'_1$ and $2w_3, 2w'_3$ satisfy one case of the Proposition 1. Or $2w_3 = 2w'_3$ and $2w_1, 2w'_1$ satisfy one case of the Proposition 1.
Let F_1 be the set of all non-entire elliptic functions in F, E_1 be the set of all non-entire elliptic functions in E, and
$\tau_1 = \{\phi, \{\{f_i\}; f_i \in F_1; i = 1,2,\cdots\}, \{\{f_i, f_j\};$
$\{\{f_i, f_j, f_k\}; f_i, f_j, f_k \in F_1; i \neq j \neq k \neq i;$
$i, j, k = 1,2,\cdots\}, \cdots, F_1, F\}$.

Lemma 3:
The topological space $(F, \cdot)$ is multiplicative group.

Proof:
From definition of F it is clear that each pair f, g of elements of F has two common periods and consequently $f \cdot g$ is doubly periodic of this two periods then $f \cdot g \in F$. Easily we can show the associative law satisfies. Since $f(z) = 1$ is in F, and $1 \cdot g = g \cdot 1 = g$, $\forall g \in F$. Also for each $f \in F$ in which f is not identically zero then $\frac{1}{f}$ is also in F and $f \cdot \frac{1}{f} = \frac{1}{f} \cdot f = 1$. Hence $(F, \cdot)$ is multiplicative group. ◊

Remark 4:
Let $\{f_m\} \in \tau_1$ then $f_m \in F_1$, and $\frac{1}{f_m} \in F_1$ consequently $\{\frac{1}{f_m}\} \in \tau_1$.

Theorem 5:
The function $i : (F, \tau_1) \to (F, \tau_1)$ defined by $i(f) = \frac{1}{f}$ is continuous.

Proof:
We must show that i is continuous function at each element of F. For this purpose we choose $f_m \in F$ arbitrary, then $i(f_m) = \frac{1}{f_m}$. Again since $f_m \in F$ so, $f_m \in F_1$ or $f_m \in F - F_1$. If $f_m \in F_1$ then by Remark (4), $\frac{1}{f_m} \in F_1$ and for the smallest τ_1-open set (neighborhood) $N_m = \{\frac{1}{f_m}\}$ of $\frac{1}{f_m}$ there exists a neighborhood $G_m = \{f_m\}$ of f_m, such that $i(G_m) = i(\{f_m\}) = \{\frac{1}{f_m}\} \subseteq \{\frac{1}{f_m}\} = N_m$ then $i(G_m) \subseteq N_m$, therefore the function i is continuous at all $f_m \in F_1$ (since f_m is arbitrary). If $f_m \notin F_1$ ($f_m \in F - F_1$), then the only τ_1-open set which contains $\frac{1}{f_m}$ is $N_m = F$ itself and also the only τ_1-open set containing f_m is $G_m = F$. Since $i(G_m) = i(F) = F \subseteq F = N_m$, then

$i(G_m) \subseteq N_m$. Therefore i is continuous at all $f_m \in F - F_1$. Hence $i:(F,\tau_1) \to (F,\tau_1)$ is continuous function. ◊

Proposition 6:

Study the continuity of the function $\omega:(F \times F,\tau_2) \to (F,\tau_1)$ on a topological space $F \times F$, in which $\omega(f,g) = f \cdot g$; $\forall (f,g) \in F \times F$, and

$\tau_2 = \{\phi, \{\{(f_i, f_j)\}; (f_i, f_j) \in F_1 \times F_1;$
$i, j = 1,2,\cdots\}, \{\{(f_i, f_j), (f_k, f_l)\};$
$(f_i, f_j), (f_k, f_l) \in F_1 \times F_1; (f_i, f_j) \neq$
$(f_k, f_l); i, j, k, l = 1,2,\cdots\}, \{\{(f_i, f_j)$
$(f_k, f_l), (f_m, f_n)\}; (f_i, f_j), (f_k, f_l),$
$(f_m, f_n) \in F_1 \times F_1; (f_i, f_j) \neq$
$(f_k, f_l) \neq (f_m, f_n) \neq (f_i, f_j); i, j, k, l,$
$m, n = 1,2,\cdots\}, \cdots, F_1 \times F_1, F \times F\}$

First we assign the space $F \times F$ by its elements which is as below $F \times F = \{(f_i, f_j); f_i, f_j \in F; i, j = 1,2,\cdots\}$ clearly each pair $(f_i, f_j) \in F \times F$ takes one of the following three forms:

1. (f_i, f_j); f_i, f_j are both entire elliptic functions in F.
2. (f_i, f_j); f_i, f_j are both non-entire elliptic functions in F.
3. (f_i, f_j) ; one of f_i and f_j is entire elliptic function and other is non-entire elliptic function in F.

It is obvious that form 2 construct the elements of $F_1 \times F_1$, because $F_1 \times F_1 = \{(f_i, f_j); f_i, f_j \in F_1; i, j = 1,2,\cdots\}$ that means $F_1 \times F_1 \subset F \times F$. Let $(f,g) \in F \times F$ then $\omega(f,g) = f \cdot g$, since $(F,\cdot)$ is multiplicative group by Lemma 4 $f \cdot g \in F$. Again since $(f,g) \in F \times F$ then either $(f,g) \in F_1 \times F_1$ or $(f,g) \in F \times F - F_1 \times F_1$. If $(f,g) \in F_1 \times F_1$ then $f \cdot g$ is doubly periodic of the two common periods of f and g and also meromorphic functions, hence $f \cdot g \in F_1$. The smallest τ_1 − open set (neighborhood) containing $f \cdot g$ is $N = \{f \cdot g\}$, and for this neighborhood there exist a τ_2 − open set $G = \{(f,g)\}$ in $F \times F$ containing (f,g) such that

$\omega(G) = \omega(\{(f,g)\}) = \{f \cdot g\} \subseteq \{f \cdot g\} = N$

then $\omega(G) \subseteq N$. Since (f,g) is arbitrary and N is the smallest τ_1 − open set containing $f \cdot g$ then ω is continuous at any point of $F_1 \times F_1$. If $(f,g) \in F \times F - F_1 \times F_1$ then (f,g) is either of the form 1 or of the form 3. Clearly if (f,g) is of the form 1, the only τ_1 − open set containing $f \cdot g$ is $N = F$ and corresponding to this neighborhood there exist an open set $G = F \times F$ in τ_2 such that $\omega(G) = \omega(F \times F) = F \subseteq F = N$, then $\omega(G) \subseteq N$. But for the case when (f,g) is of the form 3 that means one of f and g is non-entire elliptic function so is $f \cdot g$. Since for the τ_1 − open set $N = \{f \cdot g\}$ there is no τ_2 − open set G containing (f,g), such that $\omega(G) \subseteq N$ (because for every element (f_i, f_j) in τ_2 we have f_i and f_j are both non-entire elliptic functions). Hence ω is not continuous on $F \times F$. ◊

If we look at this problem precisely it'll be clear that ω is not continuous at the elements (f,g) in $F \times F$ in which f or g is entire elliptic functions. We take advantage from this outcome for construction of a topological group. For this purpose we define the two subsets M and P of F and the two topologies τ_3 on M and τ_4 on $M \times M$ as follows:

$M = F_1 \cup \{1\}$, $P = (F_1 - E_1) \cup \{1\} = M - E_1$,
$\tau_3 = \{\phi, \{\{f_i\}; f_i \in P; i = 1,2,\cdots\}, \{\{f_i, f_j\};$
$f_i, f_j \in P; i \neq j; i, j = 1,2,\cdots\},$

$\{\{f_i, f_j, f_k\}; f_i, f_j, f_k \in P; i \neq j \neq k \neq i;$
$i, j, k = 1,2,...\},..., P, M\}$, and
$\tau_4 = \{\phi, \{\{(f_i, f_j)\}; (f_i, f_j) \in P \times P;$
$i, j = 1,2,\cdots\}, \{\{(f_i, f_j), (f_k, f_l)\};$
$(f_i, f_j), (f_k, f_l) \in P \times P; (f_i, f_j) \neq (f_k, f_l);$
$i, j, k, l = 1,2,\cdots\}, \{\{(f_i, f_j)$
$(f_k, f_l), (f_m, f_n)\}; (f_i, f_j), (f_k, f_l),$
$(f_m, f_n) \in P \times P; (f_i, f_j) \neq$
$(f_k, f_l) \neq (f_m, f_n) \neq (f_i, f_j);$
$i, j, k, l, m, n = 1,2,\cdots\}, \cdots, P \times P, M \times M\}$

Theorem 7:

$(M, ., \tau_3)$ is a topological group.

Proof:

Since τ_3 is a topology on M, (M, τ_3) is a topological space, and Similar to Lemma 4 easily we can show $(M, \cdot)$ is a multiplicative group. [by the combination of the two continuity conditions, a single condition will appear which states that the function $i\omega : M \times M \to M$, given by $i\omega((f,g)) = \frac{f}{g}$, for $(f,g) \in M \times M$ is continuous]. For $(f,g) \in M \times M$, since M is multiplicative group, in which the inverse of any element f is given by $\frac{1}{f}$, so $\frac{f}{g} \in M$. Also we can say that $(f,g) \in P \times P$ or $(f,g) \in M \times M - P \times P$. For $(f,g) \in P \times P$, $\frac{f}{g} \in P$ then for the smallest $\tau_3 -$open set $N = \{\frac{f}{g}\}$ containing $\frac{f}{g}$ there is a $\tau_4 -$open set $G = \{(f,g)\}$ such that $i\omega(G) = i\omega(\{(f,g)\}) = \{\frac{f}{g}\} \subseteq \{\frac{f}{g}\} = N$ so $i\omega(G) \subseteq N$. And if $(f,g) \in M \times M - P \times P$ then either f and g are both non-entire elliptic functions in E_1 or one of them is non-entire elliptic function in E_1 and other equals to 1 (entire elliptic function). In the both cases for the only $\tau_3 -$open set $N = M$ which contains $\frac{f}{g}$ there exists a $\tau_4 -$open set $G = M \times M$ such that $i\omega(G) = i\omega(M \times M) = M \subseteq M = N$, so $i\omega(G) \subseteq N$. Hence $i\omega$ is continuous function on $M \times M$. And consequently $(M, ., \tau_3)$ is topological group. ◊

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بنيات نانی گروپێكی تەپۆلۆجی بە پشت بەستن بە نەخشە ناتەواوەكان

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بەشی ماتماتيك– كۆلێجی پەروەردەی زانستەكان– زانكۆی سەلاحەددين– هەرێمی كوردستان–عێراق

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مە بەست لەم توێژينەوەيە بنيات نانی گروپێكی تەپولوجيە ئەو بۆشايیيە وەردەگرين كە دانەكانی نەخشە ی ناتەواون (نەخشەی دوو خولين و مێرۆمۆرڤين). وە گرنگيمان داوە بە كۆمەڵەی ئەو نەخشە ناتەواوانەی كە هەمان خوليان هەيە.

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