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Boundedness of Normalized Eigenfunctions of the Spectral Problem in the Case of Weight Function Satisfying the Lipschitz Condition

Karwan H.F. Jwamer*, and Aryan Ali. M.**

Department of Mathematics, Faculty of Science and Science Education, School of Science, University of Sulaimani, Kurdistan Region, Sulaimani, Iraq.
* jwameri1973@gmail.com, ** aryanmath76@yahoo.com

Abstract:

In this paper we study the boundeness of the eigenfuctions of the spectral problem of the form:

$$-y''(x)+p_1(x)y'(x)+q_1(x)y(x)=\lambda^2\rho(x)y(x), x\epsilon(0,a) \qquad (1)$$

With the boundary conditions:

$$y(0)=0\,, y'(a)-i\lambda y(a)=0, \qquad (2)$$

and the normalized condition:

$$\left(\int_0^a \frac{\rho(x)}{e^{\int p_1(x)\,dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}=1, \qquad (3)$$

where λ is spectral parameter and $\rho(x)$ is a weight function satisfy Lipschitz condition, thatis $(\rho(x)\in Lip1)|\rho(x_2)-\rho(x_1)|\le k\,|x_2-x_1|\,,\forall\, x_1,x_2\in[0,a], k$ is Lipschitz constant, and $p_1(x)\neq 0, p_1(x)\in C^1[0,a], q_1(x)\in C[0,a]$.

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1. Introduction

Spectral analysis of Sturm-Liouville and Schrodinger differential equations with a spectralparameter in the boundary conditions has been analyzed intensively (see [1-6]). Nowadays many authors study the estimations of eigenfunctions for different spectral problems in different cases of the coefficients especially different cases of weight functions, for more details see[1-6]. In this article we study the boundeness of the problem (1)-(3) with weight function satisfy theLipschitz condition.

Let $0<m<M<\infty$ be fixed numbers such that $m<\rho(x)<M$. Here we denote $A[0,a]$ as the class of continuous functions $q_1(x)$ on $[0,a]$ that satisfy the inequality $\left|\int_{b_o}^{b_1} q_1(x)\,dx\right|<C_A$, where C_A is a constant and $[b_o,b_1]\subseteq[0,a]$.

Let us consider the countable subset $\bar{A}[0,a]=\{q_i(x)|\, i\in N\}$ of the class $A[0,a]$ satisfying the condition $\lim_{i\to\infty}\int_0^x\int_0^t q_i(s)dsdt=g_o(x)$, where $g_o(x)$ is a function that satisfies the Lipschitz condition and has uniform convergence on $[0,a]$. Let $\rho(x)\neq 1, q_1(x)\in\bar{A}[0,a], \lambda$ be acomplex number such that Im $\lambda<$ constant.

In this paper, first equation (1) with the boundary conditions (2) by using the transformation $y(x)=z(x)e^{\frac{1}{2}\int p_1(x)\,dx}$, transformed to the form:

$-z''(x)+q(x)z(x)=\lambda^2\rho(x)z(x), x\epsilon(0,a)$,(1.1)

$z(0)=0\,, z'(a)+\left(\frac{1}{2}p_1(a)-i\lambda\right)z(a)=0, (\int_0^a \rho|z(x)|^2\,dx\,)^{\frac{1}{2}}=1,$ (1.2)

where$q(x)$ is likewise continuous function on $[0,a]$, and $\rho(x)$ has not changed and $q(x)\in \bar{A}[0,a]$. And then we try to estimate the eigenfunctions of the problem (1.1)-(1.2) for sufficiently large value of $|\lambda|$ through the Cauchy problem which is defined by:

$-y''(x)+q(x)y(x)=\lambda^2\rho(x)y(x), x\epsilon(0,a)$,(1.3)

$y(0)=0, y'(0)=1$.(1.4)

2. Estimation of normalized eigenfunctions of the Cauchy problem(1.3)-(1.4):

In this section, we estimate the eigenfunction of Cauchy problem with continuous coefficients $\rho(x)$ and $q(x)$due toCauchy problem with constant coefficients $\rho(x)=\rho$ and $q(x)=q$.

As we have mentioned, at the beginning we transform equation (1) by using the transformation $y(x)=z(x)e^{\frac{1}{2}\int p_1(x)\,dx}$, and by taking the first and second derivatives of $y(x)$ with respect to x, and putting them in equation (1) give us equation (1.1) with their boundary conditions.

Theorem (1):

Let $y(x,\lambda)$ be a solution of a Cauchy problem (1.3)-(1.4), then there exist a constant k_1=$k_1(\bar{A}[0,a])$such that

$$\max_{x\in[0,a]}\frac{|y(x,\lambda)|}{(\int_0^a \rho\,|y(x,\lambda)|^2\,dx)^{\frac{1}{2}}}\leq\frac{k_1}{|\lambda|},$$

for sufficiently large value of $|\lambda|$.

Proof:

Let $y_o(x,\lambda)$ be a solution of Cauchy problem (1.3)-(1.4) with constant coefficients $\rho(x)=\rho$ and $q(x)=q$, then from [2] it was proved that:

$$y_o(x,\lambda)=\frac{-\sigma_1-i\delta_1}{\sigma_1^2+\delta_1^2}(-sinh\sigma_1x\,cos\delta_1x+icosh\sigma_1x\,sin\delta_1x),$$

and therefore:

$$y_o'(x,\lambda)=\ (cosh\sigma_1x\,cos\delta_1x-isinh\sigma_1x\,sin\delta_1x),$$

and also it was established that the following relations hold: $\sigma_1\approx\sqrt{\rho}\sigma$ and $\delta_1\approx\sqrt{\rho}\,\delta$ at large $|\lambda|$, where $\lambda=\delta_1+\mathrm{i}\sigma_1$ is an eigenvalue of Cauchy problem with a constant weight function ρ, and $\lambda=\delta+\mathrm{i}\sigma$ is eigen value of Cauchy problem where $\rho(x)$ is a function of x.

From equations $y_o(x,\lambda)$and $y_o'(x,\lambda)$ easily we can show that:

$|y_o(x,\lambda)| \le \frac{k_1}{|\lambda|}$, $|y_o{}'(x,\lambda)| \le k_o$, where k_o and k_1 are constants.

Now, we represent $y(x,\lambda)$ as a series of the form [2]:

$$y(x,\lambda) = y_o(x,\lambda) + \int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda) y_o(t_1,\lambda)\, dt_1 + \sum_{i=2}^{\infty} \int_0^x (q(t_1)-q)$$

$$y_o(x-t_1,\lambda) \int_0^{t_1} \ldots \int_0^{t_{i-1}} (q(t_1)-q)\, y_o(t_{i-1}-t_i,\lambda) y_o(t_i,\lambda) dt_i \ldots dt_1. \qquad (2.1)$$

We introduce the notations $f_i(x)$ for the term of series (2.1) with number i as follows:

$$f_1(x) = \int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda) y_o(t_1,\lambda)\, dt_1,$$

and

$$f_i(x) = \int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda) \int_0^{t_1} \ldots \int_0^{t_{i-1}} (q(t_1)-q)\, y_o(t_{i-1} - t_i,\lambda) y_o(t_i,\lambda) dt_i \ldots dt_1,$$

Then series (2.1) can be written as:

$$y(x,\lambda) - y_o(x,\lambda) = \sum_{i=1}^{\infty} f_i(x). \, (2.2)$$

By integrating equation $f_1(x)$by parts and using the condition $y_o(0,\lambda) = 0$, we obtain:

$$f_1(x) = \int_0^x \left[y_o'(x-t_1,\lambda) y_o(t_1,\lambda) - y_o(x-t_1,\lambda) y_o'(t_1,\lambda) \right]. \int_0^{t_1} (q(s)-q)\; ds\, dt_1 \qquad (2.3)$$

Differentiating the above equation with respect to x, gives:

$$f_1'(x) = y_o(x,\lambda) \int_0^x (q(s)-q) ds + \int_0^x \left(y_o''(x-t_1,\lambda) y_o(t_1,\lambda) - y_o'(x-t_1,\lambda) y_o'(t_1,\lambda) \right).$$

$$\int_0^{t_1} (q(s)-q)\; ds\, dt_1.$$

Since $y_o(x,\lambda)$ is a solution of Cauchy problem with constant coefficients ρ and q, so $y_o''(x-t_1,\lambda) = (q-\lambda^2\rho) y_o(x-t_1,\lambda)$,
therefore we deduce

$f_1'(x)$

$$= y_o(x,\lambda)\int_0^x (q(s)-q)ds + \int_0^x \left((q-\lambda^2\rho)y_o(x-t_1,\lambda)y_o(t_1,\lambda) - y_o'(x-t_1,\lambda)\right.$$

$$\left. y_o'(t_1,\lambda)\right)\int_0^{t_1} (q(s)-q)\ ds\ dt_1. \ (2.4)$$

From equation (2.3), we have

$$|f_1(x)| = \left|\int_0^x \ [y_o'(x-t_1,\lambda)y_o(t_1,\lambda) - y_o(x-t_1,\lambda)y_o'(t_1,\lambda)].\int_0^{t_1} (q(s)-q)\ ds\ dt_1\right|$$

$$\le \int_0^x |y_o'(x-t_1,\lambda)|\,|y_o(t_1,\lambda)|\left|\int_0^{t_1} (q(s)-q)\ ds\right| dt_1 + \int_0^x |y_o(x-t_1,\lambda)|\,|y_o'(t_1,\lambda)|$$

$$\left|\int_0^{t_1} (q(s)-q)\ ds\right| dt_1.$$

Since

$|y_o(x,\lambda)| \le \dfrac{k_1}{|\lambda|}$, $|y_o{}'(x,\lambda)| \le k_o$, $\forall x \in [0,a]$,

and from the choice of $A[0,a]$,

we have $\left|\int_0^{t_1} (q(s)-q)\ ds\right| < constant = k_2$,

consequently we conclude that

$|f_1(x)| \le \dfrac{k_4}{|\lambda|}$, $\forall x \in [0,a]$, where k_4 is a constant.

Again, and by the same way from equation (2.4) we get:

$$|f_1'(x)| \le \frac{k_1k_2}{|\lambda|} + \left(\left(\frac{q}{|\lambda^2|}+\rho\right)k_1^2 + k_o^2\right)k_2 a\ .$$

Now, if $i=2$ we have

$$f_2(x) = \int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda)\int_0^{t_1} (q(t_2)-q)y_o(t_1-t_2,\lambda)y_o(t_2,\lambda)dt_2dt_1$$

$$= \int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda)f_1(t_1)\ dt_1$$

Thus, $|f_2(x)| \le \frac{k_4}{|\lambda|}\left|\int_0^x (q(t_1)-q)\, y_o(x-t_1,\lambda)\ dt_1\right|$.

Integrating the last equation by parts gives:

$$|f_2(x)| \leq \frac{k_4}{|\lambda|}\left|\int_0^x y_o'(x-t_1,\lambda)dt_1\right|\left|\int_0^{t_1}(q(s)-q)\ ds\right| \leq \frac{k_2\,k_4}{|\lambda|}\int_0^x |y_o'(x-t_1,\lambda)|\,dt_1$$

or

$$|f_2(x)| \leq \frac{k_4}{|\lambda|}\ , \forall\ x \in [0,a].$$

So in a similar way, theestimations for $|f_3(x)|, |f_4(x)|, |f_5(x)| \ldots$, are carried out sequentially for $i = 2,3,4, \ldots$, thus in general

$$|f_i(x)| \leq \frac{k_{i+3}}{|\lambda|}\ , \forall\ x \in [0,a], \text{and } \forall\ i = 2,3,4, \ldots .$$

Hence for sufficiently large value of $|\lambda|$, the series in equation (2.2) converges to zero, therefore we get that

$$|y(x,\lambda)| \leq \frac{k_1}{|\lambda|}\ , \forall\ x \in [0,a], \text{and since} \left(\int_0^a \rho\,|y(x,\lambda)|^2\,dx\right)^{\frac{1}{2}} = 1,$$

this implies that

$$\max_{x\in[0,a]} \frac{|y(x,\lambda)|}{(\int_0^a \rho\,|y(x,\lambda)|^2\,dx)^{\frac{1}{2}}} \leq \frac{k_1}{|\lambda|}.$$

Hence theproof is completed.

Theorem (2):

Consider the Cauchy problem:

$-y''(x) + q(x)y(x) = \lambda^2\rho(x)y(x), x\epsilon(0,a)$, (2.5)

where$q(x)$ is a continuous function, and a weight function $\rho(x) \in Lip1$, and $\rho(a) \neq 1$. If $y(x,\lambda)$ is a solution of the Cauchy problem (2.5), then there exist a constant $\overline{d_o}$ such that

$$\max_{x\in[0,a]} \frac{|y(x,\lambda)|}{(\int_0^a \rho\,|y(x,\lambda)|^2\,dx)^{\frac{1}{2}}} \leq \frac{\overline{d_o}}{|\lambda|}\ ,$$

for sufficiently large value of $|\lambda|$.

Proof:

Let us consider the Cauchy problem (2.5) with weight function $\rho_\in(x)$ instead of $\rho(x)$, $\rho_\in(x) \in C^2[0,a]$. Now, we make the following transformations:

$$\delta(x) = \int_0^x \frac{dt}{B^2(t)} \text{and } y(x) = B(x)\,\gamma(\delta(x)), \text{where } \mathrm{B(x)} = {\rho_\in}^{-\frac{1}{4}}(x)\rho^{\frac{1}{4}}(a).$$

As a result, we shall obtain the problem:

$-\gamma''(x)+q_\epsilon(x)\gamma\big(\delta(x)\big)=\lambda^2\rho(a)\gamma\big(\delta(x)\big)\,,\delta\epsilon\ (0,b),$

with the boundary conditions

$\gamma(0)=0\,,\gamma'(0)=\sqrt[4]{\dfrac{\rho(a)}{\rho(0)}}\,,$

where $\int_0^a \frac{dt}{B^2(t)}=\frac{1}{\rho^{\frac{1}{2}}(a)}\int_0^a\sqrt{\rho(t)}\,dt=\text{constant}=b$ (not dependent on $\in$) , and

$q_\epsilon(x)=\big(q(x)B(x)-B''(x)\big)B^3(x).$

From here and up we try to show that $\left|\int_0^t q_\epsilon(\delta)d\delta\right|$ is uniform with respect to $\in$ and

$t\in[0,b]$ for small $\in>0$, (that is, we must show that $\left|\int_0^t q_\epsilon(\delta)d\delta\right|<E$, where E is a constant), then through the theorem (2.1) there exist a constant d_o such that

$\max_{\delta\in[0,b]}\frac{|\gamma(\delta)|}{\left(\int_0^b\rho(a)\,|\gamma(\delta)|^2\,d\delta\right)^{\frac{1}{2}}}\le\frac{d_o}{|\lambda|}$, from this and from the relations

$y(x)=B(x)\,\gamma(\delta(x)),\delta(x)=\int_0^x\frac{dt}{B^2(t)}$, it clearly follows that there is $\overline{d_o}$ such that

$\max_{x\in[0,a]}\frac{|y(x,\lambda)|}{(\int_0^a\rho_\in(x)\,|y(x,\lambda)|^2\,dx)^{\frac{1}{2}}}\le\frac{\overline{d_o}}{|\lambda|}$, for great value of $|\lambda|$.

Let us estimate the expression $\left|\int_0^t q_\epsilon(\delta)d\delta\right|$.

Turning to the variable x in the integral, we shall get:

$\left|\int_0^t q_\epsilon(\delta)d\delta\right|=\left|\int_0^l\big(q(x)B(x)-B''(x)\big)B^3(x)\delta'(x)dx\right|$, where $l\,\epsilon[0,a]$, and since

$\delta'(x)=\frac{1}{B^2(x)}$, hence

$$\left|\int_0^t q_\epsilon(\delta)d\delta\right| = \left|\int_0^l \left(q(x)B(x) - B''(x)\right) B(x)dx\right|$$
$$= \left|\int_0^l q(x)B^2(x)dx - \int_0^l B''(x)\, B(x)dx\right|$$
$$\le \left|\int_0^l q(x)B^2(x)dx\right| + \left|\int_0^l B''(x)\, B(x)dx\right| \le \left|\int_0^a q(x)B^2(x)dx\right| + \left|\int_0^a B''(x)\, B(x)dx\right|$$
$$\le \left|\int_0^a q(x)B^2(x)dx\right| + |B(a)B'(a)| + |B(0)B'(0)| + \left|\int_0^a \left(B'(x)\right)^2 dx\right|$$

From definition $\mathrm{B(x)} = \rho_\in(x)^{-\frac{1}{4}}\rho(a)^{\frac{1}{4}}$, it follows that

$$\mathrm{B}(0) = \sqrt[4]{\frac{\rho(a)}{\rho_\in(0)}} = \sqrt[4]{\frac{\rho(a)}{\rho(0)}}\left(\text{since from lemma}(3.4.1)[2]\rho_\in(0) = \rho(0)\right),$$

$$\mathrm{B}'(\mathrm{x}) = -\frac{\sqrt[4]{\rho(a)}\rho'_\in(x)}{4\sqrt[4]{\rho_\in^5(x)}}, \quad B'(0) = -\frac{\sqrt[4]{\rho(a)}\rho'_\in(0)}{4\sqrt[4]{\rho_\in^5(0)}}$$
$$= -\frac{\rho'_\in(0)}{4\rho(0)}\sqrt[4]{\frac{\rho(a)}{\rho(0)}}\text{ , and consequently}$$

$$\left|\int_0^t q_\epsilon(\delta)d\delta\right| \le \left|\int_0^a \sqrt{\rho(a)}\frac{q(x)}{\sqrt{\rho_\in(x)}}dx\right| + \left|\frac{\rho'_\in(a)}{4\rho(a)}\right| + \left|\frac{\rho'_\in(0)}{4\rho(0)}\sqrt{\frac{\rho(a)}{\rho(0)}}\right| +$$
$$\frac{1}{16}\int_0^a \left(\frac{\rho'_\in(x)}{\rho_\in(x)}\right)^2 \sqrt{\frac{\rho(a)}{\rho_\in(x)}}\, dx$$

Again, owing to the lemma (3.4.1) [2], $|\rho'_\in(x)| \le 2N$ and $\rho(x){-}\in \le \rho_\in(x) \le \rho(x){+}\in$,

therefore

$$\left|\int_0^t q_\epsilon(\delta)d\delta\right| \le \left|\int_0^a \sqrt{\rho(a)}\frac{q(x)}{\sqrt{\rho(x){-}\in}}dx\right| + \frac{N}{2|\rho(a)|} + \left|\frac{N}{2\rho(0)}\sqrt{\frac{\rho(a)}{\rho(0)}}\right| +$$
$$\frac{1}{4}\int_0^a \frac{N^2}{(\rho(x){-}\in)^2}\sqrt{\frac{\rho(a)}{\rho(x){-}\in}}\, dx,$$

or

$$\left|\int_0^t q_\epsilon(\delta)d\delta\right| \leq \left|\int_0^a \sqrt{\rho(a)}\frac{q(x)}{\sqrt{\rho(x)-\in}}dx\right| + c_o + c_1 + c_2\int_0^a \frac{1}{(\rho(x)-\in)^{\frac{5}{2}}}\, dx\ ,$$

where $c_o = \dfrac{N}{2|\rho(a)|}$, $c_1 = \left|\dfrac{N}{2\rho(0)}\sqrt{\dfrac{\rho(a)}{\rho(0)}}\right|$ and $c_2 = \dfrac{1}{4}N^2\sqrt{\rho(a)}$ are constants.

Since $\in$ is small, then $\left|\int_0^t q_\epsilon(\delta)d\delta\right| \leq E(\in)$, where $E(\in)$ is a constant.

Thus by theorem (1) there exist a constant d_o such that

$$\max_{\delta\in[0,b]} \frac{|\gamma(\delta)|}{\left(\int_0^b \rho(a)\,|\gamma(\delta)|^2\, d\delta\right)^{\frac{1}{2}}} \leq \frac{d_o}{|\lambda|}\ .$$

Let $\max_{\delta\in[0,b]}\left|\dfrac{1}{\sqrt{\delta'(x)}}\right| = d_1$, where $d_1$is a constant andsince $y(x) = B(x)\,\gamma\big(\delta(x)\big)$,

therefore

$$\max_{x\in[0,a]} \frac{|y(x,\lambda)|}{(\int_0^a \rho(x)\,|y(x,\lambda)|^2\, dx)^{\frac{1}{2}}} \leq \frac{\overline{d_o}}{|\lambda|}$$, where $\overline{d_o}$is a constant.

Hence the proof is achieved.

Theorem (3):

If $y(x,\lambda)$ is a solution of a Cauchy problem (2.5) and $y_o(x,\lambda)$ is a solution of a Cauchy problem with constant coefficients ρ and q, then following relations hold:

(1) $|y'(x,\lambda)| \leq k_o$, $x \in [0,a]$, where$k_o$is a constant.

(2) There is a constant C^*uniform for the whole class $A[0,a]$such that the number of local maximum points of the function $|y(x,\lambda)|$on $[0,a]$ does not exceed $C^*\sqrt{|\lambda|}$.

(3) There is a constant C_1uniform for the whole class $A[0,a]$ such that for two consecutive local maximum points $x'$and x'' of the function $|y(x,\lambda)|$, the following inequalitiesare valid:

$$\left(1+\frac{C_1}{\sqrt{|\lambda|}}\right)^{-1} < \frac{|y(x',\lambda)|}{|y(x'',\lambda)|} < 1+\frac{C_1}{\sqrt{|\lambda|}}\,.$$

Proof:

(1) Since $y(x,\lambda)$ is a solution of Cauchy problem (2.5), then

$-y''(x,\lambda)+q(x)y(x,\lambda)=\lambda^2\rho(x)y(x,\lambda), x\in(0,a)$, and $y(0,\lambda)=0$, $y'(0,\lambda)=1$.
or$q(x)y(x,\lambda)=y''(x,\lambda)+\lambda^2\rho(x)y(x,\lambda)$.
Replacing x by s , we multiply this equation by $y_o(x-s,\lambda)$and integrate from 0 to x up ds, we get:

$$\int_0^x y_o(x-s,\lambda)q(s)y(s,\lambda)\,ds \;= y_o(x-s,\lambda)y'(s,\lambda)\Big]_0^x + y_o'(x-s,\lambda)y\;(s,\lambda)\Big]_0^x + \int_0^x y_o''(x-s,\lambda)y\;(s,\lambda)ds + \int_0^x y_o(x-s,\lambda)\lambda^2\rho(s)y(s,\lambda)ds.$$

By using $y_o(0,\lambda)=1$ and $y_o'(0,\lambda)=1$ and substituting for
$y_o''(x-s,\lambda)=qy_o(x-s,\lambda)-\lambda^2\rho\, y_o(x-s,\lambda)$, we get:

$$\int_0^x y_o(x-s,\lambda)q(s)y(s,\lambda)\,ds \;= -y_o(x,\lambda)+y(x,\lambda)+\int_0^x y_o(x-s,\lambda)q\,y(s,\lambda)\,ds - \int_0^x y_o(x-s,\lambda)\lambda^2\rho\, y(s,\lambda)ds + \int_0^x y_o(x-s,\lambda)\lambda^2\rho(s)y(s,\lambda)ds$$

Or

$$y(x,\lambda)=y_o(x,\lambda)+\int_0^x (q(s)-q)\,y_o(x-s,\lambda)y(s,\lambda)\,ds - \int_0^x (\rho(s)-\rho)\lambda^2 y_o(x-s,\lambda)\;\;y(s,\lambda)ds\,.$$

We differentiate the above equation using fundamental theorem of calculus to get:
$y'(x,\lambda)=y_o'(x,\lambda)$ Or $|y'(x,\lambda)|=|y_o'(x,\lambda)|$.
But since $|y_o'(x,\lambda)|\le\; k_o$, $\forall\, x\in[0,a]$, where k_o is a constant, thus
$|y'(x,\lambda)|\le\; k_o$, $\forall\, x\in[0,a]$.
The proof of part (1) is established.

(2)Letx_o be a local maximum point of the function $|y(x,\lambda)|$ and $y_1(x,\lambda)$ be a solution of the Cauchy problem (2.5) with constant coefficients ρ and q and with the conditions $y_1(x_o,\lambda)=y\;\;(x_o,\lambda)$and $y_1'(x_o,\lambda)=y'\;(x_o,\lambda)$.
We shall take a constant c_o arbitrary and we try to estimate the difference
$|y'(x,\lambda)-y_1'(x,\lambda)|$on the interval $[x_o\,,x_o+\frac{c_o}{\sqrt{|\lambda|}}]$.
Since $y(x,\lambda)$ is a solution of Cauchy problem (2.5), then

$$-y''(x,\lambda)+q(x)y(x,\lambda)=\lambda^2\rho(x)y(x,\lambda)$$

Or

$$q(x)y(x,\lambda) = y''(x,\lambda) + \lambda^2 \rho(x) y(x,\lambda).$$

Replacing x by s , we multiply this equation by $y_1(x-s,\lambda)$and integrate from x_o to x up ds, we get:

$$\int_{x_o}^{x} y_1(x-s,\lambda) q(s) y(s,\lambda)\, ds$$

$$= y_1(x-s,\lambda) y'(s,\lambda)\Big]_{x_o}^{x} + y_1'(x-s,\lambda) y \quad (s,\lambda)\Big]_{x_o}^{x} +$$

$$\int_{x_o}^{x} y_1''(x-s,\lambda) y \quad (s,\lambda) ds + \int_{x_o}^{x} y_1(x-s,\lambda)\lambda^2 \rho(s) y(s,\lambda) ds.$$

By using the given conditions and substituting for $y_1''(x-s,\lambda) = q y_1(x-s,\lambda) - \lambda^2 \rho\, y_1(x-s,\lambda)$, we acquire:

$$y \quad (x,\lambda) = \int_{x_o}^{x} (q(s)-q) y_1(x-s,\lambda) y(s,\lambda)\, ds$$

$$- \int_{x_o}^{x} (\rho(s)-\rho)\lambda^2 y_1(x-s,\lambda) \quad y(s,\lambda) ds$$

$y_1'(x-x_o,\lambda) y_1 \ (x_o,\lambda)$.

Differentiation this equation using fundamental theorem of calculus, gives

$y' \ (x,\lambda) = y_1 \ ''(x-x_o,\lambda) \quad y_1 \ (x_o,\lambda)$.

And since $y_1 \ ''(x-x_o,\lambda) = q\ y_1 \quad (x-x_o,\lambda) - \lambda^2 \rho y_1 \quad (x-x_o,\lambda)$, therefore

$y' \ (x,\lambda) = q\ y_1 \quad (x-x_o,\lambda) y_1 \ (x_o,\lambda) - \lambda^2 \rho y_1 \quad (x-x_o,\lambda) y_1 \ (x_o,\lambda)$

Or

$y' \ (x,\lambda) - y_1'(x,\lambda) = (q-\lambda^2\rho) y_1 \quad (x-x_o,\lambda) y_1 \ (x_o,\lambda) - y_1'(x,\lambda)$.

Now

$|y' \ (x,\lambda) - y_1'(x,\lambda)| \le |\lambda^2\rho - q| \left|y_1 \quad (x-x_o,\lambda)\right| \left|y_1 \ (x_o,\lambda)\right| + |y_1'(x,\lambda)|$.

$\le \ |\lambda^2\rho| \dfrac{{k_1}^2}{|\lambda|^2} + k_1^2 \le (\rho+1)k_1^2$,

$\left(\text{because } \left|y_1 \ (x_o,\lambda)\right| \le \dfrac{k_1}{|\lambda|} \text{ and } |y_1'(x,\lambda)| \le k_o \ , \forall x \in [0,a]\right)$, hence

$|y' \ (x,\lambda) - y_1'(x,\lambda)| \le \ k^*$, where $k^* = (\rho+1)k_1^2$ is a constant.

From this inequality, integrating the difference $y' \ (x,\lambda) - y_1'(x,\lambda)$ form x_o to x with respect to x, we shall obtain:

$\left|y \quad (x,\lambda) - y_1 \ (x,\lambda)\right| \le \ \dfrac{c_1}{\sqrt{|\lambda|}}$, where $c_1$is a constant.

Due to Assertion (2.4.2) [2] not less than five extreme points of the function $|y_1\ (x\ \ ,\lambda)|$ are available for a certain c_o on the interval$[x_o\ , x_o + \frac{c_o}{\sqrt{|\lambda|}}]$.

Then through the lastestimations $|y\ \ (x,\lambda) - y_1\ (x,\lambda)| \leq \frac{c_1}{\sqrt{|\lambda|}}$, there is at least one maximum point of the function $|y_1\ (x\ \ ,\lambda)|$on the interval $[x_o\ , x_o + \frac{c_o}{\sqrt{|\lambda|}}]$.

Thus,

$$|y\ \ (x,\lambda) - y_1\ (x,\lambda)| \leq \frac{c_1}{\sqrt{|\lambda|}},$$

then

$$|y\ \ (x,\lambda)| \leq \frac{c_1}{\sqrt{|\lambda|}} + |y_1\ (x,\lambda)| \leq \frac{c^*}{\sqrt{|\lambda|}}\ ,\text{where } x \in [x_o\ , x_o + \frac{c_o}{\sqrt{|\lambda|}}], \text{and } c^* \text{ is a constant.}$$

But for the whole interval (i.e. for $x \in [0, a]$) we divide the length of the segment $[0, a]$ by $\frac{c_o}{\sqrt{|\lambda|}}$, thus the validity of the second part of our theorem hold, that is $|y\ \ (x,\lambda)| \leq C^*\sqrt{|\lambda|}$, $\forall x \in [0, a]$, where $C^*$is a constant .

(3) Let $x'$and x'' be two successive maximum points of the function $|y\ \ (x,\lambda)|$on the interval $[0, a]$, thus

$$|y\ \ (x',\lambda)| \leq C^*\sqrt{|\lambda|}\ , \text{and } |y\ \ (x'',\lambda)| \leq C^*\sqrt{|\lambda|}\ .$$

Hence, from the above inequality,followsthe validity of the proof of part three, that is

$$\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{-1} < \frac{|y(x',\lambda)|}{|y(x'',\lambda)|} < 1 + \frac{C_1}{\sqrt{|\lambda|}}.$$

Hencethe proof of theorem (3) is completed.

3. Estimation of normalized eigenfunctions of the spectral problem(1)-(3):

In this section we estimate the eigenfunctions of the problem (1)-(3) with continuous coefficients $\rho(x)$ and $q_1(x)$ due to Cauchy problem with constant coefficients ρ and q.

Theorem (4):

Let $z(x,\lambda)$ be a solution of equation (1.2) and $y_1\ (x,\lambda)$ be a solution of Cauchy problem (2.5) with constant coefficients ρ and q respectively, then there exist a constant D such that

$$\max_{x\in[0,a]} \frac{|z(x,\lambda)|}{\left(\int_0^a \rho\, |z(x,\lambda)|^2\, dx\right)^{\frac{1}{2}}} \leq D\,,$$

for sufficiently large value of $|\lambda|$and Im $\lambda <$ constant.

Proof:

Let the function $|z(x,\lambda)|$ attain the maximum at $z_m$in the point $x_o \in [0,a]$and the maximum of the function $|z'(x,\lambda)|$is equal to z'_m. Then the graph of function $|z(x,\lambda)|$ lies above a triangle with the apex at $[x_o\,, z_m\,]$, and lateral sides with angular coefficients $z'_m\,, -z'_m$consecutively. Let also $x_1\,, x_2\,, \dots\,, x_{no}$be points of maxima of the function $|z(x,\lambda)|$ lying to the right of x_o in the order of increasing (i.e. $x_1 < x_2 < \ \dots < x_{no}$), and $x_{-1}\,, x_{-2}\,, \dots\,, x_{-n1}$ be points of maxima of the function $|z(x,\lambda)|$ lying to the left of x_o in the order of decreasing (i.e. $x_{-n1} < x_{1-n1} < \cdots < x_{-2} < x_{-1}$). Similarly to point x_o, triangles inscribed under the graph of the function $|z(x,\lambda)|$ can also be formed in the points $x_{\pm i}$. Owing to selection of angles of inclination for lateral sides of triangles, bases of these triangles are not intersected (as we have shown in Figure (1)). In addition, if we cast out extreme triangles with apexes in the points $x_{-n1}$and x_{no} , then bases of the remaining triangles will obviously lie inside the segment $[0,a]$(moreover, inside the segment $[x_{-n1}, x_{no}]$) .

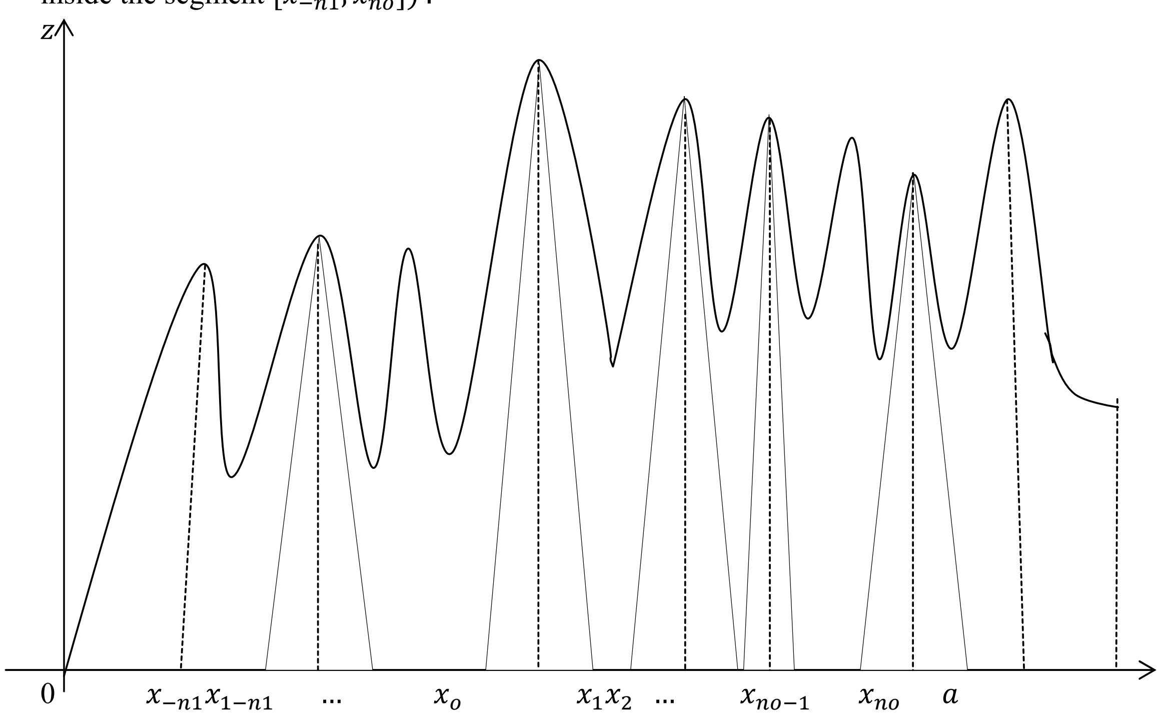

Figure (1)

Let x_i' , x_i'' be apexes lying on $[0,a]$ of the triangle, we have constructed with the apex in the point $i = 1-n_1, \dots, 0, \dots, n_o - 1$ and $z_i(x)$be a function determined on $[x_i', x_i'']$and which graph is made of triangle lateral sides.

$$Now, \int_0^a \rho|z(x,\lambda)|^2\,dx > \sum_{i=1-n_1}^{n_o-1} \int_{x_i'}^{x_i''} \rho\, z_i^2(x)\,dx \quad (3.1)$$

We try to evaluate each of the integral $\int_{x_i'}^{x_i''} \rho\, z_i^2(x)\,dx$.

First , we shall consider the integral $\int_{x_o'}^{x_o''} \rho\, z_o^2(x)\,dx$.

$$\int_{x_o'}^{x_o''} \rho\, z_o^2(x)\,dx = \int_{x_o'}^{x_o} \rho\, z_o^2(x)\,dx + \int_{x_o}^{x_o''} \rho\, z_o^2(x)\,dx = 2\rho \int_{x_o}^{x_o''} z_o^2(x)\,dx \quad (3.2)$$

The graph of $z_o(x)$ is a straight line $z_o(x) = z_o(x_o) - z_m'(x - x_o)$, on section $[x_o', x_o'']$.
Then equation (3.2) becomes:

$$\int_{x_o'}^{x_o''} \rho\, z_o^2(x)\,dx = 2\rho \int_{x_o}^{x_o''} \left(z_o(x_o) - z_m'(x - x_o)\right)^2 dx,$$ Let's find the point x_o''.

$z_o(x_o'') = z_o(x_o) - z_m'(x_o'' - x_o)$, and $z_o(x_o'') = 0$, therefore

$$x_o'' = x_o + \frac{z_o(x_o)}{z_m'}$$, thus

$$\int_{x_o'}^{x_o''} \rho\, z_o^2(x)\,dx = 2\rho \int_{x_o}^{x_o+\frac{z_o(x_o)}{z_m'}} \left(z_o(x_o) - z_m'(x - x_o)\right)^2 dx\,.$$

Let $\mu = x - x_o$, and $d\mu = dx$, so

$$\int_{x_o'}^{x_o''} \rho\, z_o^2(x)\,dx = 2\rho \int_0^{\frac{z_o(x_o)}{z_m'}} \left(z_o(x_o) - z_m'\mu\right)^2 d\mu$$

$$= 2\rho \int_0^{\frac{z_o(x_o)}{z_m'}} \left(z_o^{\ 2}(x_o) - 2z_m'\,\mu z_o(x_o) + {z_m'}^2\mu^2\right) d\mu = \frac{2\rho z_o^{\ 3}(x_o)}{3\, z_m'}.$$

Similarly, this equality is correct for other $i$values (and by the same way we can find that)

$$\int_{x_i'}^{x_i''} \rho\, z_i^2(x)\, dx = \frac{2\rho z_i^{\,3}(x_i)}{3\, z_m'}.$$

Then equation (3.1) becomes:

$$\int_0^a \rho |z(x,\lambda)|^2\, dx > \sum_{i=1-n_1}^{n_o-1} \frac{2\rho z_i^{\,3}(x_i)}{3\, z_m'} = \frac{2\rho}{3\, z_m'} \sum_{i=1-n_1}^{n_o-1} z_i^{\,3}(x_i) \qquad (3.3)$$

Then by using part (3) in theorem (2.3), there is a constant $C_1 > 0$ uniform for the whole class $A[0,a]$ such that

$$z_i \quad (x_i) > \frac{z_o \quad (x_o)}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^i}.$$

Then from equation (3.3) we have

$$\int_0^a \rho |z(x,\lambda)|^2\, dx > \frac{2\rho}{3\, z_m'} \sum_{i=1-n_1}^{n_o-1} \frac{z_o^{\,3}(x_o)}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3i}}.$$

Or

$$\int_0^a \rho |z(x,\lambda)|^2\, dx > \frac{2\rho z_o^{\,3}(x_o)}{3\, z_m'} \sum_{i=1-n_1}^{n_o-1} \frac{1}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3i}}$$

$$= \frac{2\rho z_o^{\,3}(x_o)}{3\, z_m' \left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3(1-n_1)}} \sum_{j=0}^{n_o+n_1-2} \frac{1}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3j}} \text{, where } j = i + n_1 - 1.$$

We know that $\sum_{j=0}^{n_o+n_1-2} \frac{1}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3j}}$, is a geometric series , hence we shall get

$$\int_0^a \rho |z(x,\lambda)|^2\, dx > \frac{2\rho z_o^{\,3}(x_o)}{3\, z_m'} \left(\frac{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3n_o+3n_1-3} - 1}{\left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3} - 1} \right) * \left(1 + \frac{C_1}{\sqrt{|\lambda|}}\right)^{3-3n_o}.$$

Or

$$\frac{1}{\int_0^a \rho|z(x,\lambda)|^2\,dx} < \frac{3\,z'_m}{2\rho\,{z_o}^{3}(x_o)}\left(\frac{\left(1+\frac{C_1}{\sqrt{|\lambda|}}\right)^3+1}{\left(1+\frac{C_1}{\sqrt{|\lambda|}}\right)^{3n_o+3n_1-3}+1}\right)*\left(1+\frac{C_1}{\sqrt{|\lambda|}}\right)^{3n_o+3}.$$

Now, for sufficiently large value of $|\lambda|$, we obtain

$$\frac{1}{\int_0^a \rho|z(x,\lambda)|^2\,dx} < \sqrt{\frac{3\,z'_m}{2\rho\,{z_o}^{3}(x_o)}}\,.$$

Then from this inequality, we deduce that:

$$\frac{\max\limits_{x\in[0,a]}|z(x,\lambda)|}{\left(\int_0^a \rho|z(x,\lambda)|^2\,dx\right)^{\frac{1}{2}}} \le z_o\quad(x_o)\sqrt{\frac{3\,z'_m}{2\rho\,{z_o}^{3}(x_o)}} \le \sqrt{\frac{3\,z'_m}{2\rho\,z_o\quad(x_o)}}.$$

From the choiceof z'_m , $\rho$and $z_o\quad(x_o)$ we know that all of them are positive constants, and so

is $\sqrt{\frac{3\,z'_m}{2\rho\,z_o\quad(x_o)}}$, therefore we put $D=\sqrt{\frac{3\,z'_m}{2\rho\,z_o\quad(x_o)}}>0,$ and hence

$$\max_{x\in[0,a]}\frac{|z(x,\lambda)|}{\left(\int_0^a \rho|z(x,\lambda)|^2\,dx\right)^{\frac{1}{2}}} \le D,\text{ where D is aconstant.}$$

Thus the proof of theorem is performed.

Corollary (1):

Let $y(x)$ be a solution of equation (1) in our main problem H, and $z(x)$ be a solution of equation (1.1), then there exist a constant D_1 such that

$$\max_{x\in[0,a]}\frac{|y(x)|}{\left(\int_0^a \frac{\rho}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}} \le D_1\,,$$

for sufficiently large value of $|\lambda|$ and Im $\lambda < constant$.

Proof: Since we have: $y(x)=z(x)e^{\frac{1}{2}\int p_1(x)\,dx}$, therefore

$$\frac{|y(x)|}{\left(\int_0^a \frac{\rho}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}} = \frac{z(x)e^{\frac{1}{2}\int p_1(x)\,dx}}{\left(\int_0^a \frac{\rho}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}}$$

$$= \frac{z(x)e^{\frac{1}{2}\int p_1(x)\,dx}}{\left(\int_0^a \rho\,|z(x)|^2\,dx\right)^{\frac{1}{2}}}.$$

Let $M = \max_{x\in[0,a]} e^{\frac{1}{2}\int p_1(x)\,dx}$, and since by theorem (4) $\max_{x\in[0,a]} \frac{|z(x,\lambda)|}{\left(\int_0^a \rho|z(x,\lambda)|^2\,dx\right)^{\frac{1}{2}}}$

$\leq D,$

for sufficiently large value of $|\lambda|$, where D is aconstant , thus

$$\max_{x\in[0,a]} \frac{|y(x)|}{\left(\int_0^a \frac{\rho}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}} \leq MD.$$

Let $D_1 = MD$, hence

$$\max_{x\in[0,a]} \frac{|y(x)|}{\left(\int_0^a \frac{\rho}{e^{\int p_1(x)dx}}|y(x)|^2\,dx\right)^{\frac{1}{2}}} \leq D_1$$, where D_1 is a constant.

So the proof of corollary is completed.

4. Conclusion:

In this paper, we estimate the eigenfunction of a new type of spectral problem (1) where the first derivative appear in our spectral problem (1) and the weight function satisfies the Lipschitz condition.

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