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Presentation Ideals in Locally Multiplication Modules

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Abstract

In this paper, presentation ideals in locally multiplication modules are studied. It is known that if M is a multiplication R −module and K, L are submodules of M, then $K = IM$ and $L = JM$, for ideals I, J of R and the product KL, is defined as $KL = IJM$. We generalize this concept to locally multiplication modules and we prove that under certain conditions some results that concerning presentation ideals in multiplication modules can be generalized to locally multiplication modules.

Keywords: presentation ideal, multiplication module, locally multiplication module, Primal, weakly prime, $S_M(N)$ −weakly prime.

1. Introduction

Let M be an R −module. M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$[2]. Equivalently, if for any submodule N of M, we have $N = (N: M)M$ [12], where $(N: M) = \{r \in R: rM \subseteq N\}$ is an ideal of R. M is called a locally multiplication module if M_P is a multiplication R_P −module for each maximal ideal P of R[6]. For $\phi \neq S \subseteq M$, we define $S_M(S) = \{r \in R: rs \in S, \text{for some } s \notin S\}$.
For a submodule N, we define $S_M(N) = \{r \in R: rm \in N,$ for some $m \in M - N\}$. Especially, if $N = 0$, then $S_M(0) = \{r \in R: rx = 0,$ for some $0 \neq x \in M\}$ and N is called a primal submodule of M if $S_M(N)$ forms an ideal of R[1]. A proper submodule N of M is called a prime submodule if for $r \in R, m \in M$, the condition $rm \in N$ implies that $m \in N$ or $rM \subseteq N$[12]. Moreover N is called a weakly prime submodule if whenever $0 \neq rm \in N$, where $r \in R, m \in M$, then $m \in N$ or $rM \subseteq N$[2]. Also, N is called an $S_M(N)$ −weakly prime submodule of M, if N_P is a weakly prime submodule of M_P for each maximal ideal P of R with $S_M(N) \subseteq P$[7]. A non empty subset S of R is called a multiplicative closed set if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$[9]. If S is a multiplicative closed set in R, then one can easily makes M_S as an R_S −module under the module operations $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{r}{u} . \frac{x}{s} = \frac{rx}{us}$, for $\frac{r}{u} \in R_S$ and $\frac{x}{s}, \frac{y}{t} \in M_S$[10]. Hence, when we say M_S is a module we mean M_S is an R_S −module. A submodule N is said to have a presentation ideal if $N = IM$, for some ideal I of R and I is called the presentation ideal of N[2]. If N is a submodule of M and $a, b \in M$, then by aN we mean the product $(Ra)N$ and by ab we mean the product $(Ra)(Rb)$[2]. If M is a multiplication R −module and N, K are submodules of M, then $N = IM$ and $K = JM$, for some ideals I, J of R. The product NK, of N and K, is defined as $NK = IJM$ and it is a submodule of M[2], in this case $N^2 = I^2M$.

For a submodule N, the radical of N, denoted by $\sqrt{N}$, is defined as the intersection of all prime submodules of M that contain N[2]. We define the annihilator of M as $AnnM = (0:M) = \{r \in R : rM = 0\}$. M is called finitely generated if there is a finite subset $S = \{x_1, x_2, \dots, x_n\}$ of M that generates M and in this case we write $M = < S > = < x_1, x_2, \dots, x_n >$[4]. M is called a faithful module if $AnnM = (0:M) = 0$[11]. If M is a finitely generated faithful $R-$module and S is a multiplicative closed set in R, then M_S is a finitely generated faithful R_S-module[3]. The primal spectrum of M, $pSpecM$, is defined as the set $pSpecM = \{N : N$ is a primal submodule of $M\}$[5].

2. The Main Results

Theorem 2.1. Let M be a finitely generated locally multiplication $R-$module and N is a proper submodule of M. If K and L are submodules of M and P is a maximal ideal of R such that $S_M(N) \subseteq P$, then N is a weakly prime submodule of M if and only if whenever $0 \neq K_P L_P \subseteq N_P$, then $K \subseteq N$ or $L \subseteq N$.

Proof. Let N be weakly prime and let K and L be submodules of M such that $0 \neq K_P L_P \subseteq N_P$. By [8, Proposition 2.21], N_P is a weakly prime submodule of M_P. As M is finitely generated and locally multiplication, then M_P is finitely generated multiplication. Hence, by [2, Theorem 2.6], we have $K_P \subseteq N_P$ or $L_P \subseteq N_P$. If $K_P \subseteq N_P$, then for $x \in K$, we have $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq P$, by [8, Lemma 2.1], we get $x \in N$, so that $K \subseteq N$ and if $L_P \subseteq N_P$, then by the same argument we get $L \subseteq N$. Conversely, suppose that the given condition is satisfied and we show that N_P is a weakly prime submodule of M_P. So let $0 \neq \bar{K}\bar{L} \subseteq N_P$, for the submodules $\bar{K}$ and $\bar{L}$ of M_P and then $\bar{K} = K_P$ and $\bar{L} = L_P$, for submodules K and L of M, so that $0 \neq K_P L_P \subseteq N_P$, and then by the given condition we get $K \subseteq N$ or $L \subseteq N$, which gives that $K_P \subseteq N_P$ or $L_P \subseteq N_P$ and as M_P is a finitely generated multiplication R_P-module, so by [2, Theorem 2.6], we have N_P is a weakly prime submodule of M_P and as $S_M(N) \subseteq P$, by [8, Proposition 2.21], we have N is a weakly prime submodule of M. ■

Remark 2.2. If N is a submodule of an $R-$module M, P is a maximal ideal of R and if N has a presentation ideal I, then I_P is a presentation ideal of N_P. Indeed, let $N = IM$, then clearly $N_P = (IM)_P = I_P M_P$ and then I_P is a presentation ideal of N_P.

A natural question is that, if N_P has a presentation ideal I_P, where I is an ideal of R, then under what conditions N will possess the ideal I as a presentation ideal ?.

The following result will answer this question.

Proposition 2.3. Let M be an $R-$module and P a maximal ideal of R. If N is a submodule of M such that N_P has a presentation ideal I_P, where I is an ideal of R and $S_M(N) \subseteq P, S_M(IM) \subseteq P$, then I is a presentation ideal of N.

Proof. We have $N_P = I_P M_P = (IM)_P$. Let $x \in N$, then $\frac{x}{1} \in (IM)_P$, so that $px \in IM$, for some $p \notin P$. If $x \notin IM$, then we get $p \in S_M(IM) \subseteq P$, which is a contradiction. So that $x \in IM$ and hence $N \subseteq IM$. Let $x \in IM$, then $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq P$, by [8, Lemma 2.1], we get $x \in N$. So $IM \subseteq N$ and thus $N = IM$. This means that N has a presentation ideal I. ■

By using, Remark 2.2 and Proposition 2.3, we get the following theorem.

Theorem 2.4. Let M be an $R-$module and P is a maximal ideal of R. If I is an ideal of R and N is a submodule of M such that $S_M(N) \subseteq P, S_M(IM) \subseteq P$, then I is a presentation ideal of N if and only if I_P is a presentation ideal of N_P.

Definition2.5. Let M be an $R-$module. If I is an ideal of R, then we define:

$\mathcal{C}_M = \{IM : I$ is an ideal of $R\}$ and $C = \cap \mathcal{C}_M = \cap IM$, where I is an ideal of R.

Proposition 2.6. Let R be a local ring with P as its unique maximal ideal and M be a

locally multiplication R −module. Let $\mathcal{C}_M \subseteq pSpecM$. If $K, L \in pSpecM$, then we have the following.
(1) If $aL = 0$, for all $a \in K$, then $KL = 0$.
(2) If $Kb = 0$, for all $b \in L$, then $KL = 0$.
(3)If $ab = 0$, for all $a \in K, b \in L$, then $KL = 0$.

Proof. (1) Let $\frac{a}{p} \in K_P$, for $a \in M, p \notin P$. Then $qa \in K$, for some $q \notin P$. By using [7, Corollary 2.9], we have $\frac{a}{p}L_P = (\frac{q}{q}\frac{a}{p})L_P = \frac{qa}{qp}L_P = (qaL)_P = 0$. Now, M_P is a multiplication R_P −module, so by [2, Lemma 2.5], we get $K_PL_P = 0$. Let $\bar{I}$ be a presentation ideal of K_P, so by [7, Proposition 2.16], we have $\bar{I} = I_P$, for some ideal I of R and similarly, let $\bar{J}$ be a presentation ideal of L_P, so a gain we have $\bar{J} = J_P$, for some ideal J of R and then $0 = K_PL_P = I_PJ_PM_P = (IJM)_P$. As $0 = 0M$, so $0 \in C_M$, so we get 0 is a primal submodule. Hence, $S_M(0) \subseteq P$. By [8, Lemma 2.1], we have $IJM = 0$. Since K, L, IM and JM are primal, so we have $S_M(K), S_M(L), S_M(IM)$ and $S_M(JM)$ are ideals of R and thus $S_M(K), S_M(L)$, $S_M(IM)$ and $S_M(JM)$ all are subsets of P. Since I_P is a presentation ideal of K_P and J_P is a presentation ideal of L_P, so by Theorem 2.4, I is a presentation ideal of K and J is a presentation ideal of L. This means $K = IM$ and $L = JM$, so we have $KL = IJM = 0$.
(2) It can be proved by the same technique as in (1).
(3) Let $\frac{a}{p} \in K_P$ and $\frac{b}{q} \in L_P$, for $a, b \in M$ and $p, q \notin P$. Since K, L are primal submodules, so $S_M(K), S_M(L)$ are ideals of R, so $S_M(K) \subseteq P$ and $S_M(L) \subseteq P$. Then by [8, Lemma 2.1], $a \in K$ and $b \in L$. Then $pa \in K$ and $qb \in L$. Then $\frac{a}{p}\frac{b}{q} = \frac{p}{p}\frac{a}{p}\frac{q}{q}\frac{b}{q} = \frac{pqab}{ppqq} = 0$. As M_P is a multiplication R_P −module, so by [2, Lemma 2.5] we get $K_PL_P = 0$. Moreover, $K_P = I_PM_P$ and $L_P = J_PM_P$, for ideals I_P and J_P of R_P and thus $0 = K_PL_P = I_PJ_PM_P = (IJM)_P$. Since $S_M(0) \subseteq P$, then by [8, Lemma 2.1], we get $IJM = 0$. Next, we use the same argument as in (1) to get that $KL = 0$. ■

It is known that, in a multiplication module, every submodule has a presentation ideal and now, we prove that under certain conditions this property of multiplication modules can be generalized to locally multiplication modules.

Theorem 2.7. Let M be a locally multiplication R −module and P a maximal ideal of R. If N is a proper submodule of M such that $S_M(N) \subseteq P$ and $N \subseteq C$, then N is a multiplication submodule of M.

Proof. We have M_P is a multiplication R_P −module. So that, N_P has a presentation ideal say $N_P = I_PM_P = (IM)_P$. We now show $S_M(IM) \subseteq P$. Let $r \in S_M(IM)$. Then $rx \in IM$, for some $x \notin IM$. Next, $\frac{rx}{1} \in N_P$, and by [8, Lemma 2.1], we have $rx \in N$. If $x \in N$, then as $N \subseteq C$, we have $N \subseteq IM$, so that $x \in IM$, which is a contradiction. Hence we must have $x \notin N$ and so $r \in S_M(N) \subseteq P$. So that $S_M(IM) \subseteq P$. Hence, by Proposition 2.3, I is a presentation ideal of N. ■

Theorem 2.8. Let M be a finitely generated faithful locally multiplication R −module. Let N,K be weakly prime submodules of M that are not prime with $N, K \subseteq \mathcal{M}$. If P is a maximal ideal of R such that $S_M(0)$, $S_M(N)$, $S_M(K) \subseteq P$, then $NK = 0$.

Proof. By the same argument as in Theorem 2.7, we get that $N = IM$ and $K = JM$, where $N_P = I_PM_P = (IM)_P$ and $K_P = J_PM_P = (JM)_P$with $S_M(IM) \subseteq P$ and $S_M(JM) \subseteq P$ and that $NK = IJM$. Next, we have M_P is a multiplication R_P −module, so M_P is a finitely generated faithful R_P −module. Now, as N and K are weakly prime and not prime and $S_M(N), S_M(K) \subseteq P$, so by [8, Proposition 2.21(2)] we have N_P and K_P are weakly prime which are not prime. So by [2, Corollary 2.8], we have $N_PK_P = 0$, which means that $(IJM)_P = I_PJ_PM_P = N_PK_P = 0$. As $S_M(0) \subseteq P$, so by [8, Lemma 2.1], we get $IJM = 0$ and then $NK = IJM = 0$. ■

Proposition 2.9. Let M be an R −module and P a maximal ideal of R, then $(\sqrt{0})_P \subseteq \sqrt{0_P}$.

Proof. Let $\frac{x}{p} \in (\sqrt{0})_P$, for $x \in M$ and $p \notin P$. Then $qx \in \sqrt{0}$, for some $q \notin P$. Now, let $\bar{N}$ be any prime submodule of M_P. Then by [8, Lemma 2.27], $\bar{N} = N_P$, for the prime submodule $N = \{x \in M: \frac{x}{1} \in \bar{N}\}$ and $S_M(N) \subseteq P$. So, we get $qx \in N$ and if $x \notin N$, then we get $q \in S_M(N) \subseteq P$, which is a contradiction. So we must have $x \in N$, which gives that $\frac{x}{p} \in N_P = \bar{N}$. Hence we get that $\frac{x}{p} \in \sqrt{0_P}$, that gives $(\sqrt{0})_P \subseteq \sqrt{0_P}$. ■

Proposition 2.10. Let R be a local ring with the unique maximal ideal P and M an $R-$module. Then $\sqrt{0_P} \subseteq (\sqrt{0})_P$.

Proof. Let $\frac{x}{p} \in \sqrt{0_P}$, for $x \in M$ and $p \notin P$. Let N be any prime submodule of M. If $S_M(N) \nsubseteq P$, then there exists $r \in S_M(N)$ and $r \notin P$, so that $rx \in N$ for some $x \notin N$. But as P is the unique maximal ideal of R and $r \notin P$, then r is a unit of R. Thus $x = 1.x = r^{-1}rx \in N$, which is a contradiction, so that $S_M(N) \subseteq P$. Thus by [8, Proposition 2.21], N_P is a prime submodule of M_P and thus $\frac{x}{p} \in N_P$. By [8, Lemma 2.1], we get $x \in N$ and so $x \in \sqrt{0}$, from which we get $\frac{x}{p} \in (\sqrt{0})_P$. Hence $\sqrt{0_P} \subseteq (\sqrt{0})_P$. ■

Proposition 2.11. Let R be a local ring with unique maximal ideal P. Let M be a finitely generated faithful locally multiplication $R-$module and N be a weakly prime submodule of M that is not prime with $S_M(0) \subseteq P, S_M(N) \subseteq P$. If $\mathcal{C}_M \subseteq pSpecM$, then $N\sqrt{0} = 0$.

Proof. M_P is a finitely generated faithful R_P-module and M_P is a multiplication R_P-module. As N is weakly prime but not prime and $S_M(N) \subseteq P$, by [8, Proposition 2.21], we have N_P is a weakly prime submodule of M_P but not prime. Thus by [2, Theorem 2.7], we get $N_P\sqrt{0_P} = 0$ and by using Proposition 2.9, we get $N_P(\sqrt{0})_P = 0$. Now, as M_P is a multiplication module, $N_P = I_P M_P$ and $(\sqrt{0})_P = J_P M_P$ for some ideals I and J of R. So $0 = N_P(\sqrt{0})_P = (IJM)_P$ and as $S_M(0) \subseteq P$, by [8, Lemma 2.1], $IJM = 0$. Since IM is primal, so $S_M(IM) \subseteq P$, then by Theorem 2.4, I is a presentation ideal of N and so $N = IM$. We now show that $S_M(\sqrt{0}) \subseteq P$. Let $r \in S_M(\sqrt{0})$, so $rx \in \sqrt{0}$, for some $x \notin \sqrt{0}$. Then there exists a prime submodule K of M such that $rx \in K$, which gives $r \in S_M(K)$. Since a prime submodule is primal [1], then K is a primal submodule of M and so we get $S_M(K) \subseteq P$. Thus $r \in P$. Hence $S_M(\sqrt{0}) \subseteq P$ and also as JM is primal, we have $S_M(JM) \subseteq P$ and thus by **Theorem 2.4**, J is a presentation ideal of $\sqrt{0}$. Hence $\sqrt{0} = JM$ and so $N\sqrt{0} = IJM = 0$.■

Theorem 2.12. Let R be a ring, P be a maximal ideal of R and M be a locally multiplication $R-$module. If N is a weakly prime submodule that is not prime with $S_M(N) \subseteq P$ and $S_M(IM) \subseteq P$, for any ideal I of R, then $N^2 = 0$.

Proof. As $S_M(N) \subseteq P$, by [8, Proposition 2.21], we have N_P is a weakly prime submodule of M_P that is not prime. By assumption M_P is a multiplication R_P-module, thus by [2, Corollary 2.3], we get $(N_P)^2 = 0$. Let I_P be the presentation ideal of N_P, that is $N_P = I_P M_P$. Then $(I^2M)_P = I_P I_P M_P = N_P N_P = 0$ and as $S_M(0) \subseteq P$, by [8, Lemma 2.1], we get $I^2M = 0$. Since $S_M(IM) \subseteq P$ and I_P is a presentation ideal of N_P, then by Theorem 2.4, I is a presentation ideal of N and thus $N = IM$. So we get $N^2 = NN = IIM = I^2M = 0$. ■

Theorem 2.13. Let M be a multiplication $R-$module. If N is an $S_M(N)-$weakly prime submodule that is not prime with $S_M(0) \subseteq S_M(N)$ and $\mathcal{C}_M \subseteq pSpecM$, then $N^2 = 0$.

Proof. As N is primal, $S_M(N)$ is a proper ideal of R (since $1 \notin S_M(N)$), so $S_M(N) \subseteq P$ for some maximal ideal P of R. So we have

$S_M(0) \subseteq S_M(N) \subseteq P$. As M is a multiplication module, then $N = IM$ for some ideal I of R and then $N^2 = I^2M$. Now by [8, Proposition 2.3], we have M_P is a multiplication R_P −module and as N is an $S_M(N)$ −weakly prime submodule, N_P is a weakly prime submodule of M_P. Now, if possible suppose that N_P is a prime submodule, we claim that N is a prime submodule of M. Let for $r \in R, m \in M$, $rm \in N$, then $\frac{r}{1}.\frac{m}{1} \in N_P$, so we get either $\frac{m}{1} \in N_P$ or $\frac{r}{1}M_P \subseteq N_P$. If $\frac{m}{1} \in N_P$, then by [8, Lemma 2.1], we get $m \in N$ and if $\frac{r}{1}M_P \subseteq N_P$, then by [7, Corollary 2.9], we get $(rM)_P = \frac{r}{1}M_P \subseteq N_P$. Next, for any $m \in M$, we have $\frac{r}{1}.\frac{m}{1} = \frac{rm}{1} \in N_P$ and again by [8, Lemma 2.1], we get $rm \in N$ and so we get $rM \subseteq N$, so that N is prime, which is a contradiction. Hence N_P is not a prime submodule. So by [2, Corollary 2.3], we have $(N_P)^2 = 0$. Now, $0 = (N_P)^2 = N_PN_P = I_PI_PN_P = (IIM)_P = (I^2M)_P$ and since $S_M(0) \subseteq P$, so by [8, Lemma 2.1], we get $I^2M = 0$, that is $N^2 = 0$. ■

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