Journal homepage www.jzs.uos.edu.krd
Journal of Zankoi Sulaimani
Part-A- (Pure and Applied Sciences)

On Limit Cycles of Planar Dynamical System Via Dulac-Cherkas Function

Azad Ibrahim Amen

Department of Mathematics, College of Basic Education/University of Salahaddin-Hawler-Iraq.

E-mail:azadameen1@gmail.com

Article info

Original: 06 Dec.2014
Revised:05 Jan.2015
Accepted:25.Jan.2015
Published online: 20.06.2015

Key Words:
Limit cycles
Bendixson function
Dulac-Cherkas function
planar vector field

Abstract

The main aim of this paper is to construct Bendixson-Dulac and Dulac-Cherkas functions to study the maximum number of limit cycles for several families of planar dynamical system. We also apply the results to Lienard and biochemistry reaction systems.

1.Introduction

In the qualitative theory of differential equations, research on limit cycles is an interesting and difficult part. We recall that a limit cycle is a periodic solution which has an annulus –like neighborhood free of other periodic solutions. The problem of estimating the maximum number of limit cycles for a polynomial vector field, which is the second part of the 16th Hilbert problem [1], remains unsolved yet. The more well-known method for proving the nonexistence of limit cycles is the Bendixson-Dulac method and some variation of it. We consider C^1 two dimensional autonomous system of differential equations

$$\dot{x} = P(x, y) \ , \ \ \dot{y} = Q(x, y) \qquad (1)$$

defined on an open subset U of $\mathbb{R}^2$ and their corresponding vector fields $\chi = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ on U and the divergence of χ is $div\,\chi = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

2. Preliminaries.

In this section we present the basic definitions and results necessary to prove some theorems and corollaries.

Theorem 1.(Bendixson-Dulac)[2].If there exists a continuously differentiable function $B(x, y)$ in a simply connected region U such that $div(\,B\chi)$ has constant sign and is not identically zero in any sub region , then the system (1) does not possess any limit cycle(closed trajectory) which lies entirely in U.

The function $B(x,y)$ is called Dulac function of the system (1).In general it is very difficult to find a Dulac function for system (1).Theorem1 can be extended to multiply connected regions in order to study the nonexistence and existence of limit cycles of system (1).
Theorem 2 [7].Let U be a ℓ - connected open subsets of $\mathbb{R}^2$ with smooth boundary .If B Dulac function defined in U, then the system (1) has at most $\ell - 1$ limit cycles which lie entirely in U.

The method of Dulac function was generalized by Cherkas in [2].The corresponding generalized Dulac function are called Dulac-Cherkas function and are defined as follows.

Definition [2].Let $P, Q \in C^1(U, \mathbb{R})$in some open region $U \subset \mathbb{R}^2$. A function $\Psi \in C^1(U, \mathbb{R})$ is called a Dulac-Cherkas function of system (1) in U if there exist a real number $\kappa \neq 0$ such that

$$\Phi := (grad\,\Psi, \chi) + \kappa\Psi\, div\chi > 0 (< 0) \quad \text{in} \quad U \tag{2}$$

Theorem 3[2].Let U be a $\ell -$ connected region andΨ be a Dulac-Cherkas function of system (1) in U such that $\mathcal{W} = \{(x,y) \in U: \Psi(x,y) = 0\}$ has s ovals in U.Then system (1) has at most$\ell - 1 + s$limit cycles in U,any existing limit cycle is hyperbolic.

In [2, 4] are considered the generalized Lienard system of the form

$$\dot{x} = y, \quad \dot{y} = \sum_{j=0}^{l} h_j(x)\, y^j \text{ with } l \geq 1 .$$

for such systems constructed a Dulac-Cherkas function. See [3, 6] for details about constructions of Bendixson-Dulac and Dulac-Cherkas functions .

3. The main results.

In this section, we discuss the nonexistence and existence of limit cycles of planar differential systems by using the qualitative theory of ordinary differential equations.

Theorem 4.Suppose that D is a simply connected domain and if either $h'(x)$ or $h'(x) + h_1(x)$ does not change sign and vanishes at most on a set of measure zero in D. Then the system

$$\dot{x} = f(y) + g(y)h(x) =: P(x,y) \quad , \quad \dot{y} = g(y)h_1(x) =: Q(x,y) \tag{3}$$

has no limit cycles in D, where $f, g, h, h_1 \in C^1(D)$.

Proof. First construct the Dulac function $B_1(x,y) = \frac{1}{g(y)}$ .Along solutions of system (3), we derive that $div(B_1P, B_1Q) = h'(x)$. Second construct the Dulac function $B_2(x,y) = \frac{e^y}{g(y)}$, we have

$div(B_2P, B_2Q) = e^y(h_1(x) + h'(x))$. According to the hypothesis, $div(B_iP, B_iQ)$ is constant sign in D for $i = 1,2$.The result follows by Theorem 1.■

Theorem 5.Suppose that D be a $\ell -$ connected region ($\ell \geq 1$) in $\mathbb{R}^2$ and if either $h'(x)$ or $h'(x) + h_1(x)$ does not change sign and vanishes at most on a set of measure zero in D. Then the system (3) has at most $\ell - 1$ limit cycles which lie entirely in D.

Proof. Since D be a $\ell -$ connected region, then the result follows from Theorem 2 and Theorem 4. ■

Using the same technique of proof as that in Theorem 4, we prove the following result.

Theorem 6.Suppose that D is a simply connected domain and if either $g^{'}(y)$ or $g^{'}(y) + h(y)$ does not change sign and vanishes at most on a set of measure zero in D, then the system

$$\dot{x} = f(x)\, h(y) \ , \quad \dot{y} = g(y)f(x) + h_1(x)$$

has no limit cycles in D, where $f, g, h, h_1 \in C^1(D)$.

Theorem 7.The system $\dot{x} = f_1(x)\big(y^{n-1} + f(y)\big) =: P(x,y)$,

$$\dot{y} = g(x) + (ay^{n-1} + b\, y^n f(x) + n\, b\, F(y))f_1(x) =: Q(x,y) \tag{4}$$

has no limit cycles, where $F(y) = \int f(y)dy$, $f_1(0) = 0$, $n \in \mathbb{R}\backslash\{1\}$ and a, b are real parameters.

Proof. First if $a \neq 0$, we take the Dulac function $B(x,y) = \frac{a\, e^{-nbx}}{f_1(x)}$.Then it follows that

$div(BP, BQ) = a^2(n-1)y^{n-2}e^{-nbx}$ is a constant sign in the region $y > 0$ or $y < 0$.The *x*–axis is invariant because $\dot{x} = 0$ on $x = 0$,hence there are no limit cycles in the whole plane by Bendixson-Dulac Theorem. Now if $a = 0$, multiply the vector field by a function $B(x,y) = \frac{e^{-nbx}}{f_1(x)}$, then phase portrait of system (3) is essentially unchanged if we multiply the vector field by a non-zero function. Since$div(BP, BQ) = 0$, then the system is exact hence the system (4) has no limit cycles.■

We use the same proof as in Theorem 7, to obtain the following result.

Theorem 8.The system

$$\dot{x} = g(y) + (ax^{n-1} + b\, x^n + n\, b\, F(x))f_1(y) \ ,$$

$$\dot{y} = (x^{n-1} + f(x))f_1(y)$$

has no limit cycles, where $F(x) = \int f(x)\, dx$, $f_1(0) = 0$, $n \in \mathbb{R}\backslash\{1\}$ and a, b are real parameters.

Corollary 1.The following systems

$$\dot{x} = f_1(x)(y^{n-1} + a_0 + a_1 y + a_2 y^2 + \cdots + a_k y^k) \ ,$$

$$\dot{y} = g(x) + (ay^{n-1} + b\, y^n + nb(a_0 y + \tfrac{a_1 y^2}{2} + \cdots + \tfrac{a_k y^k}{k+1} +)f_1(x) \qquad \text{and}$$

$$\dot{x} = g(y) + (ax^{n-1} + b\, x^n + nb(a_0 x + \frac{a_1 x^2}{2} + \cdots + \frac{a_k x^k}{k+1})f_1(y)$$

$$\dot{y} = (x^{n-1} + a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k)f_1(y)$$

has no limit cycles, where $f_1(0) = 0$, $k \in \mathbb{R}\backslash\{1\}$ and a, b, a_i are real parameters.

Proof. We take $f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_k y^k$ and $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$. Apply Theorem 7 and Theorem 8, the corollary follows. ■

In this paper we also consider the system

$$\dot{x} = h(y)h^{'}(y) =: P(x)$$

(5)

$$\dot{y} = -h(y)f(x) - g(x) =: Q(x)$$

where $f, g, h \in C^1$ and $F(x) = \int f(x)\,dx$ and $G(x) = \int g(x)\,dx$ are a primitive functions of f and g respectively

Theorem 9.Suppose that D is a simply connected domain and if either $h^{'}(y)[g(x)\,F(x) - 2f(x)G(x)]$ or $h^{'}(y)f(x)([h^2(y) - 2G(x)]$ does not change sign and vanishes at most on a set of measure zero in D. Then the system (5) has no limit cycles in D.

Proof. First construct the Dulac function $B_1(x,y) = \frac{1}{y^2+yF(x)+2G(x)}$ .Along solutions of system (5), we derive that $div(B_1P, B_1Q) = \frac{h^{'}(y)[g(x)\,F(x)-2f(x)G(x)]}{{B_1}^2(x,y)}$.Second construct the Dulac function $B_2(x,y) = \frac{1}{h^2(y)+2G(x)}$ .Then we have $div(B_2P, B_2Q) = \frac{h^{'}(y)f(x)[h^2(y)-2G(x)]}{{B_2}^2(x,y)}$. The conditions imply that $div(B_iP, B_iQ)$ is constant sign in D for $i = 1,2$.The result follows by Theorem 1.■

In [8], the author discussed Lienard's equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \qquad (6)$$

where f and g are real polynomials satisfying the conditions $f \neq 0$ and deg(f) ≥ deg(g),and proves the equation (6) has no algebraic closed solutions. Equation (5) is equivalent to the system

$$\dot{x} = y \ , \ \dot{y} = -yf(x) - g(x) \qquad (7)$$

which is a special case of system(5) when $h(y) = y$.It is easy to derive the following result by Theorem 9.

Corollary 2. Suppose that D is a simply connected domain and if either $g(x)F(x) - 2f(x)G(x) > 0\ (< 0)$ or $(x)[y^2 - 2G(x)] > 0 (< 0)$, then the system (7) has no limit cycles in D.

Note that if $f(x)$ is a constant sign in a simply connected region D, then the system (7) has no limit cycles in D.

Now we use Dulac-Cherkas function to study limit cycles for a generalized Biochemistry reaction systems

$$\dot{x} = a - bx - xf(y)$$

(8)

$$\dot{y} = bx - y + xf(y)$$

It was studied in [7], the following Biochemistry reaction model

$$\dot{x} = a - bx - xy^n$$

(9)

$$\dot{y} = bx - y + xy^n$$

where $y \geq 0$, $n$is positive integer and a $> 0, b \geq 0$ and obtained some sufficient conditions for the boundedness of solutions and the existence of limit cycles of (9).Clearly, system (8) contains of the model (9) when $f(y) = y^n$.

Theorem 10.Let U be a $\ell(\ell \geq 1)$ connected region in $\mathbb{R}^2$. If $1 + 4yf'(y) < 0$,then the system (8)has at most $\ell - 1$limit cycles in U, any existing limit cycle is hyperbolic.

Proof. We construct a Dulac-Cherkas function in the form

$$\Psi(y) = b + f(y) \qquad (10)$$

The curve $\Psi(y) = 0$ has no closed curves. Moreover, from (2) and (10) we get

$$\Phi(x,y) = -\kappa f^2(y) + ((\kappa + 1)\big(xf'(y) - 2\kappa b - \kappa\big)f(y) + (bx - y - \kappa bx)f'(y) - \kappa b(b + 1).$$

Setting $\kappa = -1$, we get $\Phi(x,y) = [(f(y) + \left(b + \frac{1}{2}\right)]^2 - \frac{1}{4} - yf'(y)$

Thus Φ does not depend on x .According to the hypothesis, then $\Phi(x,y) > 0$ and the result follows by Theorem 3.■

Corollary 3.Let U be a $\ell(\ell \geq 1)$ connected region in $\mathbb{R}^2$.If $nb > \frac{(n-1)^2}{4}$, then the system (9) has at most $\ell - 1$ limit cycles in U, any existing limit cycle is hyperbolic.

Proof. By recursive calculations, and proceeding in the proof of Theorem 10 by taking $\Psi(y) = b + y^n$ then the corollary follows. ■

Corollary 4. Let U be a simply connected region in $\mathbb{R}^2$.

1. If $1 + 4yf'(y) < 0$, then the system (8) has no limit cycles in U.

2. If $nb > \frac{(n-1)^2}{4}$, then the system (9) has no limit cycles in U.

Proof. Because U be a simply connected region, hence corollary follows directly from Theorem 10, Corollary 3 and Theorem 1. ■

In [6], studied the differential system

$$\dot{x} = f_1(x) + f_2(y) =: P(x,y)$$

(11)

$$\dot{y} = x\, g(y) =: Q(x,y)$$

where f_1, f_2 and g areC^1 functions .For system(11), Gine has constructed a Dulac function of the form $B = c\, g(y)$, where $c$is an arbitrary constant. But in fact B is not a Dulac function for system (11) if $f_1'(x) > 0$ or$f_1'(x) < 0$because $div(BP, BQ) = c_2\, g(y)\big[f_1'(x) + 2x\, g'(y)\big]$.Nowwe construct a Dulac function for a general system of (11).

Theorem 11.Consider the differential system

$$\dot{x} = f_1(x) + f_2(y) =: P(x,y)$$

(12)

$$\dot{y} = h(x)\, g(y) =: Q(x, y)$$

If $\frac{f_1{'}(x)}{g(y)} > 0$ or $\frac{f_1{'}(x)}{g(y)} < 0$ in a simply connected region D, then system (12) has no limit cycles in D.

Proof. We choose a Dulac function in the form $\mathrm{B(x, y)} = \frac{1}{g(y)}$.The divergence of the vector field $(\mathrm{BP.BQ})$ equals $\frac{f_1{'}(x)}{g(y)}$.According to the hypothesis then $\mathrm{div(BP.BQ)}$ is a constant and is not identically zero in D and the result follows by Theorem 1.■

Conclusion.

There is no general method for determining an appropriate Dulac and Dulac –Cherkas function for a given system (1). In this work we apply the Bendixson-DulacTheorem and Dulac-Cherkas method to give an upper bound of the number of limit cycles for several families of planar systems.

References

[1]Chavarriga J. and Grau M. Some open problems related to 16th Hilbert problem. Series A: Mathematical Sciences, Vol. 9 (2003), 1–26.

[2]Cherkas L. ,Grin A., and Schneider K.R.Dulac-Cherkas functions for generalizedLienard systems. Elec.J. Qual.Theo.Diff.Eqs.35(2011)1-23.

[3]Cherkas L. and GrinA..Bendixson-Dulac criterion and reducation to global uniqueness in the problem of estimating the number of limit cycles.Diff,Eqs. vo.46,no.1(2010)61-69.

[4]Cherkas L. and GrinA..On a Dulac function for the Kuklessystem.Diff,Eqs. vo.46,no.6(2010)818-826.

[5]Gasull A. andGiacominiH..Some applications of the extended Bendixson-Dulactheorem.preprint, arXiv: 1035.3402(2013).

[6]GineJ.Dulac functions of planar vector fields.Qual.Theo.Dyn.Sys.vol.13 (2014)121-128.

[7]LiuX.G.Limit cycles in a class of biochemistry reaction systems. Dyn. Cont.Disc. Imp.Sys.A.Math. Ana.12(2005)675-684.

[8]Odani K., The integration of polynomial Li´enard systems by elementary functions, Diff. Eqs.Dyn. Sys 5(3-4) (1997) 347–354, Planar nonlinear dynamical systems (Delft, 1995).