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Solving Coupled Hirota System by Using Homotopy Perturbation and Homotopy Analysis Methods

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Abstract

In this paper, two methods, namely Homotopy Perturbation Method (**HPM**) and Homotopy Analysis Method (**HAM**) are applied to obtain approximate solutions of the nonlinear coupled Hirota system (**CHS**). We see that these two methods are efficient and effectives for solving nonlinear CHS and the obtained results of the two methods coincide with each other. In our work, Maple 13 has been used for computations.

Introduction

The nonlinear phenomena played a very significant role in the field of applied Mathematics and mathematical Physics. Since in the presence of computer programming software's, the solution of a linear equation is not a problem. But to solve nonlinear problems analytically, it is still difficult for the mathematicians. The analytical methods are fastly developing, but still have some deficiencies and shortcomings, which do not satisfy the mathematician. It is well known that many phenomena in scientific fields can be described by nonlinear partial differential equations. The nonlinear models of real-life problems are still difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods in determining a solution, approximate or exact, analytical or numerical, to nonlinear models ***[2, 4, 8]***.

To describe nonlinear CHS, we start with Hirota equation ***[1, 5]***:

$$\frac{\partial w}{\partial t}+3\alpha|w|^2\frac{\partial w}{\partial x}+\gamma\frac{\partial^3 w}{\partial x^3}=0, -\infty<x<\infty, t>0, \qquad (1)$$

where w is a complex valued function of the spatial coordinate x and the time t, α and γ are positive real constants. This equation is an integrable equation which has a number of physical applications, such as the propagation of optical pluses in nematic liquid crystal waveguides. The Hirota equation is closely related to both the nonlinear Schrodinger (NLS) and modified Korteweg-de Vries (mKdV) equations, as it is complex generalization of the mKdV equation and it is a part of the NLS hierarchy of the integrable equation. Also, its soliton solution has a very similar form to the NLS soliton. The Hirota equation (1) has a two-parameter soliton family, with amplitude and velocity. The exact solution of Hitora equation (1) is

$$w(x,t) = \beta \operatorname{sech}[k(x-Vt)] \exp(i\varphi),$$
$$\beta = \sqrt{\frac{2\gamma}{\alpha}}\, k \qquad , \quad \varphi = a(x-bt), \tag{2}$$
$$V = \gamma(k^2 - 3a^2) \ , \ \ b = \gamma(3k^2 - a^2).$$

β is the amplitude of the wave, k is related to the width of the wave envelope and V is the velocity ***[5]***. The parameter a is the wave number of the phase and b is related to the frequency of the phase. Also the solution is $x = x_0$ at $t = 0$ .The Hirota equation has the conserved quantities

$$I_1 = \int_{-\infty}^{\infty} |w|^2 dx = \text{constant}\,,$$
$$I_2 = \int_{-\infty}^{\infty} \left(\frac{\alpha}{2}|w|^4 - |w_x|^2\right) dx = \text{constant}. \tag{3}$$

The Hirota equation (1) has been solved analytically by sine-cosine and tanh methods by Wazwaz ***[9]*** and showed that this equation admits sech-shaped soliton solutions whose amplitudes and velocities are free parameters, and tanh solution (kink type). Also solved by ***[8]*** by tanh method. Hirota method also used by ***[4]*** for solving (1). To avoid complex computation which needs too many calculations in the solution of (1), we assume

$$w(x,t) = u(x,t) + iv(x,t), \qquad i^2 = -1$$

where $u(x,t)$ and $v(x,t)$ are real functions. After calculations, this will reduce Hirota equation (1) to the coupled Hirota system (CHS)

$$\frac{\partial u}{\partial t} + 3\alpha f(u,v)\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3} = 0,$$
$$\frac{\partial v}{\partial t} + 3\alpha f(u,v)\frac{\partial v}{\partial x} + \gamma\frac{\partial^3 v}{\partial x^3} = 0. \tag{4}$$

where $f(u,v) = u^2 + v^2$. In this paper, we try to solve (4) numerically by HPM and HAM. Like other nonlinear analytical technique, HPM and HAM are two well-known methods for obtaining approximate solutions to the differential equations.

Basic Idea of the Homotopy Perturbation Method

To illustrate the basic idea of this method ***[3]***, we consider the following general nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{5}$$

with the following boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{6}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be decomposed into a linear part and a non-linear one, designated as L and N respectively. Hence Equation (5) can be written as the following form:

$$L(u) + N(u) - f(r) = 0.$$

Using Homotopy technique, we construct a Homotopy, $v(r,p) : \Omega \times [0,1] \rightarrow R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of Equation (5), which satisfies the boundary conditions (5).

Obviously, from the Equation (7), we have:

$$H(v,0) = L(v) - L(u_0) = 0, \quad \text{and} \quad H(v,1) = A(v) - f(r) = 0.$$

By changing the value of p from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$, in topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called Homotopic. Due to the fact that

$p \in [0,1]$ can be considered as a small parameter, hence we consider the solution of Equation (7), as a power series in p as the following:

$$v = v_0 + v_1 p + v_2 p^2 + \cdots. \quad (8)$$

In Equation (8), setting $p = 1$, results approximate solution of the problem (5)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots , \quad (9)$$

which is the approximate solution for (1) see ***[2]***.

Applying Homotopy Perturbation Method to Solve the Coupled Hirota System

For solving CHS (4) by HPM. we construct the following Homotopy:
$U(r_1, p_1)$ and $V(r_2, p_2)$ from the Cartesian set : $\Omega \times [0,1]$ into R which satisfies:

$$H(U,V,p_1) = (1-p_1)[L(U) - L(u_0)] + p_1[A_1(U,V) - f(r_1)] = 0,$$
$$H(U,V,p_2) = (1-p_2)[L(V) - L(v_0)] + p_2[A_2(U,V) - f(r_2)] = 0,$$

where

$$U = U_0 + U_1 p_1 + U_2 p_1^2 + \cdots, \quad (10)$$
$$V = V_0 + V_1 p_2 + V_2 p_2^2 + \cdots. \quad (11)$$

$$L(U) = \frac{\partial U}{\partial t}, L(u_0) = \frac{\partial u_0}{\partial t}, u_0 = u(x,0) \text{ and } L(V) = \frac{\partial V}{\partial t}, L(v_0) = \frac{\partial v_0}{\partial t}, v_0 = v(x,0).$$

That is, from Equation (7), we obtain:

$$H(U,V,p_1) = (1-p_1)\left[\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t}\right] + p_1\left[\frac{\partial U}{\partial t} + 3\alpha(U^2+V^2)\frac{\partial U}{\partial x} + \gamma\frac{\partial^3 U}{\partial x^3}\right] = 0,$$

and

$$H(U,V,p_2) = (1-p_2)\left[\frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t}\right] + p_2\left[\frac{\partial V}{\partial t} + 3\alpha(U^2+V^2)\frac{\partial V}{\partial x} + \gamma\frac{\partial^3 V}{\partial x^3}\right] = 0.$$

Hence, we have:

$$H(U,V,p_1) = \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p_1\left[\frac{\partial u_0}{\partial t} + 3\alpha(U^2+V^2)\frac{\partial U}{\partial x} + \gamma\frac{\partial^3 U}{\partial x^3}\right] = 0, \quad (12)$$

and

$$H(U,V,p_2) = \frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t} + p_2\left[\frac{\partial v_0}{\partial t} + 3\alpha(U^2+V^2)\frac{\partial V}{\partial x} + \gamma\frac{\partial^3 V}{\partial x^3}\right] = 0. \quad (13)$$

Substituting the Equations (10) and (11), into the Equations (12) and (13), respectively, we obtain

$$\begin{aligned} H(U,V,p_1) = {} & \frac{\partial}{\partial t}(U_0 + U_1 p_1 + U_2 p_1^2 + \cdots) - \frac{\partial u_0}{\partial t} \\ & + p_1\Big[\frac{\partial u_0}{\partial t} + 3\alpha[(U_0 + U_1 p_1 + U_2 p_1^2 + \cdots)^2 \\ & + (V_0 + V_1 p_2 + V_2 p_2^2 + \cdots)^2]\frac{\partial}{\partial x}(U_0 + U_1 p_1 + U_2 p_1^2 + \cdots) \\ & + \gamma\frac{\partial^3}{\partial x^3}(U_0 + U_1 p_1 + U_2 p_1^2 + \cdots)\Big] = 0, \end{aligned} \quad (14)$$

and

$$\begin{aligned} H(U,V,p_2) = {} & \frac{\partial}{\partial t}(V_0 + V_1 p_2 + V_2 p_2^2 + \cdots) - \frac{\partial v_0}{\partial t} \\ & + p_2\Big[\frac{\partial v_0}{\partial t} + 3\alpha[(U_0 + U_1 p_1 + U_2 p_1^2 + \cdots)^2 \\ & + (V_0 + V_1 p_2 + V_2 p_2^2 + \cdots)^2]\frac{\partial}{\partial x}(V_0 + V_1 p_2 + V_2 p_2^2 + \cdots) \end{aligned}$$

$$+\gamma\frac{\partial^3}{\partial x^3}(V_0+V_1p_2+V_2p_2^2+\cdots)\Bigg]=0. \quad (15)$$

In Equations (14) and (15), equating the terms with identical powers of p_1 and p_2, respectively, leads to:

$$p_1^0: \frac{\partial U_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, \qquad U_0(x,t)=u_0(x,t), \quad (16)$$

$$p_2^0: \frac{\partial V_0}{\partial t}-\frac{\partial v_0}{\partial t}=0, \qquad V_0(x,t)=v_0(x,t), \quad (17)$$

$$p_1^1: \frac{\partial U_1}{\partial t}+\frac{\partial u_0}{\partial t}+3\alpha[U_0^2+V_0^2]\frac{\partial U_0}{\partial x}+\gamma\frac{\partial^3 U_0}{\partial x^3}=0, \qquad U_1(x,0)=0, \quad (18)$$

$$p_2^1: \frac{\partial V_1}{\partial t}+\frac{\partial v_0}{\partial t}+3\alpha[U_0^2+V_0^2]\frac{\partial V_0}{\partial x}+\gamma\frac{\partial^3 V_0}{\partial x^3}=0, \qquad V_1(x,0)=0, \quad (19)$$

$$p_1^2: \frac{\partial U_2}{\partial t}+6\alpha[U_0U_1+V_0V_1]\frac{\partial U_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial U_1}{\partial x}+\gamma\frac{\partial^3 U_1}{\partial x^3}=0, \; U_2(x,0)=0, \quad (20)$$

$$p_2^2: \frac{\partial V_2}{\partial t}+6\alpha[U_0U_1+V_0V_1]\frac{\partial V_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial V_1}{\partial x}+\gamma\frac{\partial^3 V_1}{\partial x^3}=0, \; V_2(x,0)=0. \quad (21)$$

$$\vdots$$

So we can find $U_0, V_0, U_1, V_1, \ldots$ from the Equations (16)-(21) by applying the inverse operator on the interval $(0,t)$,

$$L_t^{-1}=\int_0^t(\cdot)\,dt,$$

As follows:

From the Equations (16) and (17), we get:

$U_0(x,t)=u_0(x,t)$, and $V_0(x,t)=v_0(x,t)$.

From the Equations (18) and (19) respectively, since $U_0(x,t)$ and $V_0(x,t)$ are initial solutions depends on x only, and $\frac{\partial U_0}{\partial t}=\frac{\partial V_0}{\partial t}=0$, we get:

$$\begin{aligned}U_1(x,t)&=-\int_0^t\left(\frac{\partial U_0}{\partial t}+3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial U_0(x,t)}{\partial x}+\gamma\frac{\partial^3 U_0(x,t)}{\partial x^3}\right)dt\\&=-\left(3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial U_0(x,t)}{\partial x}+\gamma\frac{\partial^3 U_0(x,t)}{\partial x^3}\right)t,\\&=w_{10}(x,t)t,\end{aligned}$$

$$\begin{aligned}V_1(x,t)&=-\int_0^t\left(\frac{\partial V_0}{\partial t}+3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial V_0(x,t)}{\partial x}+\gamma\frac{\partial^3 V_0(x,t)}{\partial x^3}\right)dt\\&=-\left(3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial V_0(x,t)}{\partial x}+\gamma\frac{\partial^3 V_0(x,t)}{\partial x^3}\right)t\\&=w_{20}(x,t)t,\end{aligned}$$

where

$$w_{10}(x,t)=-\left(3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial U_0(x,t)}{\partial x}+\gamma\frac{\partial^3 U_0(x,t)}{\partial x^3}\right),$$

and

$$w_{20}(x,t)=-\left(3\alpha[U_0^2(x,t)+V_0^2(x,t)]\frac{\partial V_0(x,t)}{\partial x}+\gamma\frac{\partial^3 V_0(x,t)}{\partial x^3}\right).$$

From the Equations (20) and (21), we get:

$$U_2(x,t)=-\int_0^t\left(6\alpha[U_0U_1+V_0V_1]\frac{\partial U_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial U_1}{\partial x}+\gamma\frac{\partial^3 U_1}{\partial x^3}\right)dt,$$

$$V_2(x,t)=-\int_0^t\left(6\alpha[U_0U_1+V_0V_1]\frac{\partial V_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial V_1}{\partial x}+\gamma\frac{\partial^3 V_1}{\partial x^3}\right)dt.$$

Substituting the values of $U_1(x,t)$ and $V_1(x,t)$ into the above two equations of $U_2(x,t)$ and $V_2(x,t)$, yields:

$$U_2(x,t) = -\int_0^t \left(6\alpha[U_0(w_{10}(x,t)t)+V_0(w_{20}(x,t)t)]\frac{\partial U_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}(w_{10}(x,t)t) + \gamma\frac{\partial^3}{\partial x^3}(w_{10}(x,t)t)\right)dt,$$

$$= -\left(6\alpha[U_0 w_{10}(x,t)+V_0 w_{20}(x,t)]\frac{\partial U_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}w_{10}(x,t) + \gamma\frac{\partial^3}{\partial x^3}w_{10}(x,t)\right)\left(\frac{t^2}{2}\right)$$

$$= w_{11}(x,t)\left(\frac{t^2}{2}\right),$$

$$V_2(x,t) = -\int_0^t \left(6\alpha[U_0 w_{10}(x,t)t+V_0 w_{20}(x,t)t]\frac{\partial V_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}(w_{20}(x,t)t) + \gamma\frac{\partial^3}{\partial x^3}(w_{20}(x,t)t)\right)dt$$

$$= -\left(6\alpha[U_0 w_{10}(x,t)+V_0 w_{20}(x,t)]\frac{\partial V_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}w_{20}(x,t) + \gamma\frac{\partial^3}{\partial x^3}w_{20}(x,t)\right)\left(\frac{t^2}{2}\right)$$

$$= w_{21}(x,t)\left(\frac{t^2}{2}\right).$$

where

$$w_{11}(x,t) = -\left(6\alpha[U_0 w_{10}(x,t)+V_0 w_{20}(x,t)]\frac{\partial U_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}w_{10}(x,t) + \gamma\frac{\partial^3}{\partial x^3}w_{10}(x,t)\right),$$

and

$$w_{21}(x,t) = -\left(6\alpha[U_0 w_{10}(x,t)+V_0 w_{20}(x,t)]\frac{\partial V_0}{\partial x} + 3\alpha[U_0^2+V_0^2]\frac{\partial}{\partial x}w_{20}(x,t) + \gamma\frac{\partial^3}{\partial x^3}w_{20}(x,t)\right),$$

and so on. Substituting these values of $U_0, V_0, U_1, V_1, U_2, V_2, \ldots$ into the Equations (10) and (11), we get:

$$u(x,t) = \lim_{p_1\to 1} U(x,t) = \lim_{p_1\to 1}(U_0 + U_1 p_1^1 + U_2 p_1^2 + \cdots) = U_0 + U_1 + U_2 + \cdots,$$

$$v(x,t) = \lim_{p_2\to 1} V(x,t) = \lim_{p_2\to 1}(V_0 + V_1 p_2^1 + V_2 p_2^2 + \cdots) = V_0 + V_1 + V_2 + \cdots.$$

which are the approximate solutions of the CHS.

Basic Idea of the Homotopy Analysis Method

In this section, according to HAM ***[6, 7]***, consider the following differential equation:

$$N[u(\tau)] = 0, \tag{22}$$

where N is a nonlinear operator, τ denotes independent variables, $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the Homotopy method ***[7]***, constructs the so called zero-order deformation equation

$$(1-p)L[\varphi(\tau;p) - u_0(\tau)] = phH(\tau)N[\varphi(\tau;p)], \tag{23}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a nonzero parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$is an initial guess of $u(\tau)$, $\varphi(\tau;p)$ is a unknown function. It is important, that one has great freedom to choose auxiliary things in HAM.

Obviously, when $p = 0$ and $p = 1$ it holds, $\varphi(\tau;0) = u_0(\tau)$ and $\varphi(\tau;1) = u(\tau)$, respectively. Thus, as p increases from 0 to 1, the solution $\varphi(\tau;p)$ varies from the initial guesses $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\varphi(\tau;p)$ in Taylor series with respect to p, we have:

$$\varphi(\tau;p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau)p^m,$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \varphi(\tau;p)}{\partial p^m}\bigg|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function $H(\tau)$ are so properly chosen, the above series converges at $p = 1$, and then we have:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau).$$

Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), u_2(\tau), \dots, u_n(\tau)\}.$$

Differentiating Equation (23), m times with respect to p, then setting $p = 0$ and finally dividing them by $m!$, we obtain the following m^{th} -order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau)R_m(\vec{u}_{m-1}), \qquad (24)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\varphi(\tau;p)}{\partial p^{m-1}}\bigg|_{p=0},$$

and

$$\chi_m = \begin{cases} 0\,, & m \le 1 \\ 1\,, & m > 1. \end{cases}$$

Applying $L^{-1} = \int_0^t (\cdot)dt$ to both sides of (24), we get

$$u_m(\tau) = \chi_m u_{m-1}(\tau) + hL^{-1}[H(\tau)R_m(\vec{u}_{m-1})].$$

In this way, it is easily to obtain u_m for $m \ge 1$, at m^{th}- order, we have:

$$u(\tau) = \sum_{m=0}^{M} u_m(\tau).$$

When $M \to \infty$ we get an accurate approximation of the original Equation (22). For the convergence of the above method we refer the reader Liao's work. If Equation (22) admits unique solution, then this method will produce the unique solution. If Equation (22) does not possess unique solution, the HAM will give a solution among many other (possible) solutions ***[10].***

Applying Homotopy Analysis Method to Solve the Coupled Hirota System

In order to find the solution of CHS (4) by using HAM, we choose the following auxiliary linear operators:

$$L[\varphi_1(x,t;p_1)] = \frac{\partial}{\partial t}[\varphi_1(x,t;p_1)],$$
$$L[\varphi_2(x,t;p_2)] = \frac{\partial}{\partial t}[\varphi_2(x,t;p_2)],$$

which satisfy

$$L[C_1 + tC_2] = 0,$$

where $C_1(x)$ and $C_2(x)$ are the integral constants. Now, we define nonlinear operators as follows:

$$N_1[\varphi_1,\varphi_2] = \frac{\partial\varphi_1(x,t;p_1)}{\partial t} + 3\alpha\left[\big(\varphi_1(x,t;p_1)\big)^2\,\frac{\partial\varphi_1(x,t;p_1)}{\partial x} + \big(\varphi_2(x,t;p_2)\big)^2\frac{\partial\varphi_1(x,t;p_1)}{\partial x}\right] + \gamma\frac{\partial^3\varphi_1(x,t;p_1)}{\partial x^3},$$

and

$$N_2[\varphi_1,\varphi_2] = \frac{\partial \varphi_2(x,t;p_2)}{\partial t} + 3\alpha\left[\left(\varphi_1(x,t;p_1)\right)^2 \frac{\partial \varphi_2(x,t;p_2)}{\partial x} + \left(\varphi_2(x,t;p_2)\right)^2 \frac{\partial \varphi_2(x,t;p_2)}{\partial x}\right] + \gamma \frac{\partial^3 \varphi_2(x,t;p_2)}{\partial x^3}.$$

Using the above definition, we construct the zeroth-order deformation equation:

$$(1-p_1)L[\varphi_1(x,t;p_1) - u_0(x,t)] = p_1 hH(x,t;p_1)N_1[\varphi_1,\varphi_2], \tag{25}$$

for $p_1 = 0$ and $p_1 = 1$, we can write

$$\varphi_1(x,t;0) = u_0(x,t), \qquad \varphi_1(x,t;1) = u(x,t), \tag{26}$$

$$(1-p_2)L[\varphi_2(x,t;p_2) - v_0(x,t)] = p_2 hH(x,t;p_2)N_2[\varphi_1,\varphi_2]. \tag{27}$$

For $p_2 = 0$ and $p_2 = 1$, we can write

$$\varphi_2(x,t;0) = v_0(x,t), \qquad \varphi_2(x,t;1) = v(x,t). \tag{28}$$

As p_1 and p_2 increases from 0 to 1, $\varphi_1(x,t;p_1)$ and $\varphi_2(x,t;p_2)$ vary from $u_0(x,t)$ and $v_0(x,t)$ to the exact solutions $u(x,t)$ and $v(x,t)$. Due to Taylor's theorem, Equations (26) and (28), we can express

$$\varphi_1(x,t;p_1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p_1^m, \tag{29}$$

$$\varphi_2(x,t;p_2) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)p_2^m, \tag{30}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi_1(x,t;p_1)}{\partial p_1^m}\bigg|_{p_1=0}, \quad v_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi_2(x,t;p_2)}{\partial p_2^m}\bigg|_{p_2=0}.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameters h are so properly chosen, series (29) and (30) are convergent, at $p_1 = 1$ and $p_2 = 1$, then one has:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{31}$$

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t). \tag{32}$$

Now, we define the vectors

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), u_2(x,t), \dots, u_n(x,t)\},$$
$$\vec{v}_n = \{v_0(x,t), v_1(x,t), v_2(x,t), \dots, v_n(x,t)\}.$$

Differentiating the zeroth-order deformation Equations (25) and (27) m times with respect to p_1 and p_2, then dividing by $m!$, and finally setting $p_1 = 0$ and $p_2 = 0$, we get the following m^{th}-order deformation equations:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)R_{1m}(\vec{u}_{m-1},\vec{v}_{m-1}), \tag{33}$$

$$L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = hH(x,t)R_{2m}(\vec{u}_{m-1},\vec{v}_{m-1}), \tag{34}$$

with the initial conditions

$$u_m(x,0) = 0, \qquad v_m(x,0) = 0,$$

where

$$R_{1m}(\vec{u}_{m-1},\vec{v}_{m-1}) = \frac{\partial u_{m-1}(x,t)}{\partial t} + 3\alpha\left(\sum_{n=0}^{m-1}\sum_{s=0}^{n}[u_s(x,t)u_{n-s}(x,t) + v_s(x,t)v_{n-s}(x,t)]\frac{\partial u_{m-1-n}(x,t)}{\partial x}\right) + \gamma\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}, \tag{35}$$

$$R_{2m}(\vec{u}_{m-1},\vec{v}_{m-1}) = \frac{\partial v_{m-1}(x,t)}{\partial t} + 3\alpha\left(\sum_{n=0}^{m-1}\sum_{s=0}^{n}[u_s(x,t)u_{n-s}(x,t)\right.$$

$$+v_s(x,t)v_{n-s}(x,t)]\frac{\partial v_{m-1-n}(x,t)}{\partial x}\Bigg)+\gamma\frac{\partial^3 v_{m-1}(x,t)}{\partial x^3}. \tag{36}$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1.\end{cases}$$

Now, the solution of the m^{th}-order deformation Equations (33) and (34) for $m \geq 1$ and $H(x,t)=1$ becomes:

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+hL^{-1}[R_{1m}(\vec{u}_{m-1}(x,t),\vec{v}_{m-1}(x,t))], \tag{37}$$

$$v_m(x,t)=\chi_m v_{m-1}(x,t)+hL^{-1}[R_{2m}(\vec{u}_{m-1}(x,t),\vec{v}_{m-1}(x,t))]. \tag{38}$$

In the following parts, we consider the initial approximations and determine other components of the solution series for the CHS.

Numerical Applications

In this section, we will apply HPM and HAM to solve the nonlinear CHS, and present numerical results to verify the effectiveness of this method, we take the following example:

Example 1:

Consider the following nonlinear CHS:

$$u_t+3\alpha(u^2+v^2)u_x+\gamma u_{xxx}=0,$$
$$v_t+3\alpha(u^2+v^2)v_x+\gamma v_{xxx}=0,$$

with the initial conditions

$$u_0(x,t)=\sqrt{\frac{2\gamma}{\alpha}}\,k\,\mathrm{sech}(kx)\cos(ax),$$

and

$$v_0(x,t)=\sqrt{\frac{2\gamma}{\alpha}}\,k\,\mathrm{sech}(kx)\sin(ax),$$

where k and a are arbitrary constants.

Note: The exact solutions of the Hirota equation (1) is given by (2), where α and $\gamma>0$, are arbitrary constants.

Solution:

(a) Using HPM:

Apply the HPM (see Section 3, Equations (16)-(21)), and equating the terms with the identical powers of p_1 and p_2 as:

$$p_1^0:\frac{\partial U_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, \qquad U_0(x,t)=u_0(x,t),$$

$$p_2^0:\frac{\partial V_0}{\partial t}-\frac{\partial v_0}{\partial t}=0, \qquad V_0(x,t)=v_0(x,t),$$

$$p_1^1:\frac{\partial U_1}{\partial t}+\frac{\partial u_0}{\partial t}+3\alpha[U_0^2+V_0^2]\frac{\partial U_0}{\partial x}+\gamma\frac{\partial^3 U_0}{\partial x^3}=0, \qquad U_1(x,0)=0, \tag{39}$$

$$p_2^1:\frac{\partial V_1}{\partial t}+\frac{\partial v_0}{\partial t}+3\alpha[U_0^2+V_0^2]\frac{\partial V_0}{\partial x}+\gamma\frac{\partial^3 V_0}{\partial x^3}=0, \qquad V_1(x,0)=0, \tag{40}$$

$$p_1^2:\frac{\partial U_2}{\partial t}+6\alpha[U_0U_1+V_0V_1]\frac{\partial U_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial U_1}{\partial x}+\gamma\frac{\partial^3 U_1}{\partial x^3}=0, \quad U_2(x,0)=0, \tag{41}$$

$$p_2^2:\frac{\partial V_2}{\partial t}+6\alpha[U_0U_1+V_0V_1]\frac{\partial V_0}{\partial x}+3\alpha[U_0^2+V_0^2]\frac{\partial V_1}{\partial x}+\gamma\frac{\partial^3 V_1}{\partial x^3}=0, \quad V_2(x,0)=0. \tag{42}$$

Then solving Equations (39)-(42), we obtained U_1, V_1, U_2 and V_2 as follows:

$$U_1 = \frac{\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{\cosh(kx)^2}[(k^3 - 3a^2k)\sinh(kx)\cos(ax) + (3ak^2 - a^3)\sin(ax)\cosh(kx)],$$

$$V_1 = \frac{\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{\cosh(kx)^2}[(k^3 - 3a^2k)\sinh(kx)\sin(ax) + (a^3 - 3ak^2)\cos(ax)\cosh(kx)],$$

$$U_2 = \frac{-t^2\gamma^{\frac{5}{2}}\, k\sqrt{\frac{2}{\alpha}}}{2\cosh(kx)^3}[(-k^6 + a^6 + 15a^2k^4 - 15a^4k^2)\cosh(kx)^2\cos(ax) + (20a^3k^3 - 6ak^5 - 6a^5k)\sinh(kx)\sin(ax)\cosh(kx) + (2k^6 - 12a^2k^4 + 18a^4k^2)\cos(ax)],$$

$$V_2 = \frac{-t^2\gamma^{\frac{5}{2}}\, k\sqrt{\frac{2}{\alpha}}}{2\cosh(kx)^3}[(15a^2k^4 - 15a^4k^2 - k^6 + a^6)\cosh(kx)^2\sin(ax) + (6ak^5 + 6a^5k - 20a^3k^3)\cos(ax)\cosh(kx)\sinh(kx) + (18a^4k^2 - 12a^2k^4 + 2k^6)\sin(ax)].$$

Then the approximate solutions of second-order are

$$u_2(x,t) = \lim_{p_1\to 1}(U_0 + p_1U_1 + p_1^2U_2) = U_0 + U_1 + U_2,$$
$$v_2(x,t) = \lim_{p_2\to 1}(V_0 + p_2V_1 + p_2^2V_2) = V_0 + V_1 + V_2.$$

(b) Using HAM:

Appling HAM (see Section 5), by using Equations (35)-(38) with the initial conditions $u_0(x,t)$ and $v_0(x,t)$, we have:

$$u_1(x,t) = \frac{h\,\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{\cosh(kx)^2}[(-k^3 + 3a^2k)\sinh(kx)\cos(ax) + (a^3 - 3ak^2)\sin(ax)\cosh(kx)],$$

$$v_1(x,t) = \frac{h\,\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{\cosh(kx)^2}[(-k^3 + 3a^2k)\sinh(kx)\sin(ax) + (3ak^2 - a^3)\cos(ax)\cosh(kx)],$$

$$u_2(x,t) = \frac{-h\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{2\cosh(kx)^3}[(2k^3 - 6a^2k)\sinh(kx)\cos(ax)\cosh(kx) + (h\gamma ta^6 - h\gamma tk^6 - 15h\gamma ta^4k^2 + 15h\gamma ta^2k^4)\cosh(kx)^2\cos(ax) + (20h\gamma tk^3a^3 - 6h\gamma ta^5k - 6h\gamma tak^5)\sinh(kx)\sin(ax)\cosh(kx) + (6ak^2 - 2a^3 + 6hak^2 - 2ha^3)\cosh(kx)^2\sin(ax) + (2hk^3 - 6ha^2k)\sinh(kx)\cos(ax)\cosh(kx) + (2h\gamma tk^6 + 18h\gamma ta^4k^2 - 12h\gamma ta^2k^4)\cos(ax)],$$

$$v_2(x,t) = \frac{-h\gamma^{\frac{3}{2}}\, tk\sqrt{\frac{2}{\alpha}}}{2cosh(kx)^3}[(2a^3 - 6ak^2 - 6hak^2 + 2ha^3)\cosh(kx)^2\cos(ax) + (6h\gamma tak^5 - 20h\gamma ta^3k^3 + 6h\gamma ta^5k)\sinh(kx)\cos(ax)\cosh(kx) + (h\gamma ta^6 - h\gamma tk^6 + 15h\gamma ta^2k^4 - 15h\gamma ta^4k^2)\sin(ax)\cosh(kx)^2 + (2hk^3 - 6a^2k + 2k^3 - 6ha^2k)\sinh(kx)\sin(ax)\cosh(kx) + (18h\gamma ta^4k^2 + 2h\gamma tk^6 - 12h\gamma ta^2k^4)\sin(ax)].$$

Then the approximate solutions of second-order are:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t),$$
$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t).$$

Note: The results obtained by HPM and HAM for Example 1, are tabulated in the tables listed below followed by their figures for $\alpha = 2, \gamma = 1, k = 0.2$ and $a = 0.5$.

Table (1) Comparison of the exact solution of $u(x,t)$ with the approximate solution obtained by HAM for different value of h, $-4 \le x \le 4$ and $0 \le t \le 1$.

Time t	Exact Solution	HAM $h = -0.9$	HAM $h = -1$	HAM $h = -1.1$
		$x = -4$		
0	-0.06223059112	-0.06223059114	-0.06223059114	-0.06223059114
0.2	-0.06159459550	-0.06196413673	-0.06159492463	-0.06195992879
0.4	-0.06087357030	-0.06168064017	-0.06087614031	-0.06166380842
0.6	-0.06006578734	-0.06138010147	-0.06007423819	-0.06134223002
0.8	-0.05916973998	-0.06106252061	-0.05918921827	-0.06099519360
1	-0.05818416660	-0.06072789761	-0.05822108054	-0.06062269916
		$x = -2$		
0	0.09995673184	0.09995673184	0.09995673184	0.09995673184
0.2	0.1030424592	0.1013004885	0.1030437931	0.1012989412
0.4	0.1060895804	0.1026379788	0.1061002925	0.1026317899
0.6	0.1090899599	0.1039692029	0.1091262302	0.1039552779
0.8	0.1120354076	0.1052941607	0.1121216060	0.1052694051
1	0.1149177307	0.1066128522	0.1150864199	0.1065741715
		$x = 0$		
0	0.2	0.2	0.2	0.2
0.2	0.1999024782	0.1999799975	0.199902444	0.1999701197
0.4	0.1996103218	0.1999199900	0.199609776	0.1998804789
0.6	0.1991247547	0.1998199775	0.199121996	0.1997310775
0.8	0.1984478028	0.1996799600	0.198439104	0.1995219156
1	0.1975822747	0.1994999375	0.197561100	0.1992529931
		$x = 2$		
0	0.09995673184	0.09995673184	0.09995673184	0.09995673184
0.2	0.09684043014	0.09860670893	0.09683910874	0.09860516170
0.4	0.09370143634	0.09725041973	0.09369092381	0.09724423082
0.6	0.09054744092	0.09588786426	0.09051217705	0.09587393920
0.8	0.08738590422	0.09451904252	0.08730286847	0.09449428685
1	0.08422402234	0.09314395449	0.08406299805	0.09310527376
		$x = 4$		
0	-0.06223059112	-0.06223059114	-0.06223059114	-0.06223059114
0.2	-0.06278348354	-0.06248000340	-0.06278313985	-0.06247579546
0.4	-0.06325537422	-0.06271237351	-0.06325257075	-0.06269554176
0.6	-0.06364851734	-0.06292770147	-0.06363888385	-0.06288983004
0.8	-0.06396529804	-0.06312598729	-0.06394207915	-0.06305866029
1	-0.06420821168	-0.06330723095	-0.06416215664	-0.06320203251

Table (2) Comparison of the exact solution of $v(x,t)$ with the approximate solution obtained by HAM for different value of h, $-4 \leq x \leq 4$ and $0 \leq t \leq 1$.

Time t	Exact Solution	HAM $h=-0.9$	HAM $h=-1$	HAM $h=-1.1$
		$x=-4$		
0	-0.1359763223	-0.1359763224	-0.1359763224	-0.1359763224
0.2	-0.1393458350	-0.1374403051	-0.1393469390	-0.1374400307
0.4	-0.1427031908	-0.1389031765	-0.1427121354	-0.1389020789
0.6	-0.1460413479	-0.1403649366	-0.1460719113	-0.1403624669
0.8	-0.1493529467	-0.1418255852	-0.1494262669	-0.1418211947
1	-0.1526303286	-0.1432851225	-0.1527752023	-0.1432782623
		$x=-2$		
0	-0.1556733863	-0.1556733863	-0.1556733863	-0.1556733863
0.2	-0.1559812729	-0.1558237946	-0.1559819203	-0.1558165216
0.4	-0.1561417698	-0.1559447473	-0.1561467943	-0.1559156555
0.6	-0.1561515918	-0.1560362446	-0.1561680082	-0.1559707880
0.8	-0.1560079740	-0.1560982864	-0.1560455620	-0.1559819190
1	-0.1557087083	-0.1561308727	-0.1557794557	-0.1559490486
		$x=0$		
0	0	0	0	0
0.2	0.002598878622	0.001128796634	0.0026000000	0.001128796635
0.4	0.005191038134	0.002257593267	0.0052000000	0.002257593270
0.6	0.007769805124	0.003386389900	0.0078000000	0.003386389906
0.8	0.010328596930	0.004515186535	0.0104000000	0.004515186541
1	0.012860965490	0.005643983168	0.0130000000	0.005643983176
		$x=2$		
0	0.1556733863	0.1556733863	0.1556733863	0.1556733863
0.2	0.1552218762	0.1554935225	0.1552211923	0.1554862496
0.4	0.1546309478	0.1552842033	0.1546253381	0.1552551114
0.6	0.1539052008	0.1550454285	0.1538858240	0.1549799719
0.8	0.1530495848	0.1547771983	0.1530026498	0.1546608309
1	0.1520693535	0.1544795126	0.1519758155	0.1542976885
		$x=4$		
0	0.1359763223	0.1359763224	0.1359763224	0.1359763224
0.2	0.1326013605	0.1345112283	0.1326002854	0.1345109539
0.4	0.1292273116	0.1330450228	0.1292188282	0.1330439252
0.6	0.1258601838	0.1315777060	0.1258319507	0.1315752363
0.8	0.1225056265	0.1301092778	0.1224396527	0.1301048872
1	0.1191689280	0.1286397382	0.1190419345	0.1286328780

Table (3) Absolute errors for the results in Table (1).

Space x	Time t	$\|\text{Exact} - \text{HAM}\|$ $h = -0.9$	$\|\text{Exact} - \text{HAM}\|$ $h = -1$	$\|\text{Exact} - \text{HAM}\|$ $h = -1.1$
-4	0	2.0000E-11	2.0000E-11	2.0000E-11
	0.2	3.6954E-04	3.2913E-07	3.6533E-04
	0.4	8.0707E-04	2.5700E-06	7.9024E-04
	0.6	1.3143E-03	8.4508E-06	1.2764E-03
	0.8	1.8928E-03	1.9478E-05	1.8255E-03
	1	2.5437E-03	3.6914E-05	2.4385E-03
-2	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	1.7420E-03	1.3339E-06	1.7435E-03
	0.4	3.4516E-03	1.0712E-05	3.4578E-03
	0.6	5.1208E-03	3.6270E-05	5.1347E-03
	0.8	6.7412E-03	8.6198E-05	6.7660E-03
	1	8.3049E-03	1.6869E-04	8.3436E-03
0	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	7.7519E-05	3.4200E-08	6.7641E-05
	0.4	3.0967E-04	5.4580E-07	2.7016E-04
	0.6	6.9522E-04	2.7587E-06	6.0632E-04
	0.8	1.2322E-03	8.6988E-06	1.0741E-03
	1	1.9177E-03	2.1175E-05	1.6707E-03
2	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	1.7663E-03	1.3214E-06	1.7647E-03
	0.4	3.5490E-03	1.0513E-05	3.5428E-03
	0.6	5.3404E-03	3.5264E-05	5.3265E-03
	0.8	7.1331E-03	8.3036E-05	7.1084E-03
	1	8.9199E-03	1.6102E-04	8.8813E-03
4	0	2.0000E-11	2.0000E-11	2.0000E-11
	0.2	3.0348E-04	3.4369E-07	3.0769E-04
	0.4	5.4300E-04	2.8035E-06	5.5983E-04
	0.6	7.2082E-04	9.6335E-06	7.5869E-04
	0.8	8.3931E-04	2.3219E-05	9.0664E-04
	1	9.0098E-04	4.6055E-05	1.0062E-03

Table (4) Absolute errors for the results in Table (2).

Space x	Time t	$\lvert\text{Exact} - \text{HAM}\rvert$ $h = -0.9$	$\lvert\text{Exact} - \text{HAM}\rvert$ $h = -1$	$\lvert\text{Exact} - \text{HAM}\rvert$ $h = -1.1$
-4	0	1.0000E-10	1.0000E-10	1.0000E-10
	0.2	1.9055E-03	1.1040E-06	1.9058E-03
	0.4	3.8000E-03	8.9446E-06	3.8011E-03
	0.6	5.6764E-03	3.0563E-05	5.6789E-03
	0.8	7.5274E-03	7.3320E-05	7.5318E-03
	1	9.3452E-03	1.4487E-04	9.3521E-03
-2	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	1.5748E-04	6.4740E-07	1.6475E-04
	0.4	1.9702E-04	5.0245E-06	2.2611E-04
	0.6	1.1535E-04	1.6416E-05	1.8080E-04
	0.8	9.0312E-05	3.7588E-05	2.6055E-05
	1	4.2216E-04	7.0747E-05	2.4034E-04
0	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	1.4712E-03	1.1214E-06	1.4701E-03
	0.4	2.9424E-03	8.9619E-06	2.9334E-03
	0.6	4.4136E-03	3.0195E-05	4.3834E-03
	0.8	5.8848E-03	7.1403E-05	5.8134E-03
	1	7.3560E-03	1.3903E-04	7.2170E-03
2	0	0.0000E+00	0.0000E+00	0.0000E+00
	0.2	2.7165E-04	6.8390E-07	2.6437E-04
	0.4	6.5326E-04	5.6097E-06	6.2416E-04
	0.6	1.1402E-03	1.9377E-05	1.0748E-03
	0.8	1.7276E-03	4.6935E-05	1.6112E-03
	1	2.4102E-03	9.3538E-05	2.2283E-03
4	0	1.00000E-10	1.00000E-10	1.00000E-10
	0.2	1.90987E-03	1.07510E-06	1.90959E-03
	0.4	3.81771E-03	8.48340E-06	3.81661E-03
	0.6	5.71752E-03	2.82331E-05	5.71505E-03
	0.8	7.60365E-03	6.59738E-05	7.59926E-03
	1	9.47081E-03	1.26993E-04	9.46395E-03

Table (5) Running times for calculating the approximate values of $u(x,t)$ with two iterations

Time t	Exact Solution	Approximate Solution of the HPM	Approximate Solution of the HAM	Least Square Error of HPM	Least Square Error of HAM
			$x=-4$		
0	-0.06223059112	-0.06223059114	-0.06223059114	1.8202E-09	1.8202E-09
0.2	-0.06159459550	-0.06159492463	-0.06159492463		
0.4	-0.06087357030	-0.06087614031	-0.06087614031		
0.6	-0.06006578734	-0.06007423819	-0.06007423819		
0.8	-0.05916973998	-0.05918921827	-0.05918921827		
1	-0.05818416660	-0.05822108054	-0.05822108054		
			$x=-2$		
0	0.09995673184	0.09995673184	0.09995673184	3.7318E-08	3.7318E-08
0.2	0.10304245920	0.10304379310	0.10304379310		
0.4	0.10608958040	0.10610029250	0.10610029250		
0.6	0.10908995990	0.10912623020	0.10912623020		
0.8	0.11203540760	0.11212160600	0.11212160600		
1	0.11491773070	0.11508641990	0.11508641990		
			$x=0$		
0	0.2	0.2	0.2	5.3195E-10	5.3195E-10
0.2	0.1999024782	0.199902444	0.199902444		
0.4	0.1996103218	0.199609776	0.199609776		
0.6	0.1991247547	0.199121996	0.199121996		
0.8	0.1984478028	0.198439104	0.198439104		
1	0.1975822747	0.197561100	0.197561100		
			$x=2$		
0	0.099956731840	0.09995673184	0.09995673184	3.4180E-08	3.4180E-08
0.2	0.096840430140	0.09683910874	0.09683910874		
0.4	0.093701436340	0.09369092381	0.09369092381		
0.6	0.090547440920	0.09051217705	0.09051217705		
0.8	0.087385904220	0.08730286847	0.08730286847		
1	0.084224022340	0.08406299805	0.08406299805		
			$x=4$		
0	-0.06223059112	-0.06223059114	-0.06223059114	2.7610E-09	2.7610E-09
0.2	-0.06278348354	-0.06278313985	-0.06278313985		
0.4	-0.06325537422	-0.06325257075	-0.06325257075		
0.6	-0.06364851734	-0.06363888385	-0.06363888385		
0.8	-0.06396529804	-0.06394207915	-0.06394207915		
1	-0.06420821168	-0.06416215664	-0.06416215664		

Table (5) Running times for calculating the approximate values of $v(x,t)$ with two iterations

Time t	Exact Solution	Approximate Solution of the HPM	Approximate Solution of the HAM	Least Square Error of HPM	Least Square Error of HAM
			$x = -4$		
0	-0.1359763223	-0.1359763224	-0.1359763224	2.7380E-08	2.7380E-08
0.2	-0.1393458350	-0.1393469390	-0.1393469390		
0.4	-0.1427031908	-0.1427121354	-0.1427121354		
0.6	-0.1460413479	-0.1460719113	-0.1460719113		
0.8	-0.1493529467	-0.1494262669	-0.1494262669		
1	-0.1526303286	-0.1527752023	-0.1527752023		
			$x = -2$		
0	-0.1556733863	-0.1556733863	-0.1556733863	6.7132E-09	6.7132E-09
0.2	-0.1559812729	-0.1559819203	-0.1559819203		
0.4	-0.1561417698	-0.1561467943	-0.1561467943		
0.6	-0.1561515918	-0.1561680082	-0.1561680082		
0.8	-0.1560079740	-0.1560455620	-0.1560455620		
1	-0.1557087083	-0.1557794557	-0.1557794557		
			$x = 0$		
0	0	0	0	2.5422E-08	2.5422E-08
0.2	0.00259887862	0.0026000000	0.0026000000		
0.4	0.00519103813	0.0052000000	0.0052000000		
0.6	0.00776980512	0.0078000000	0.0078000000		
0.8	0.01032859693	0.0104000000	0.0104000000		
1	0.01286096549	0.0130000000	0.0130000000		
			$x = 2$		
0	0.1556733863	0.1556733863	0.1556733863	1.1360E-08	1.1360E-08
0.2	0.1552218762	0.1552211923	0.1552211923		
0.4	0.1546309478	0.1546253381	0.1546253381		
0.6	0.1539052008	0.1538858240	0.1538858240		
0.8	0.1530495848	0.1530026498	0.1530026498		
1	0.1520693535	0.1519758155	0.1519758155		
			$x = 4$		
0	0.1359763223	0.1359763224	0.1359763224	2.1350E-08	2.1350E-08
0.2	0.1326013605	0.1326002854	0.1326002854		
0.4	0.1292273116	0.1292188282	0.1292188282		
0.6	0.1258601838	0.1258319507	0.1258319507		
0.8	0.1225056265	0.1224396527	0.1224396527		
1	0.1191689280	0.1190419345	0.1190419345		

One Picture worth more than ten thousand words (*Anonymous*)

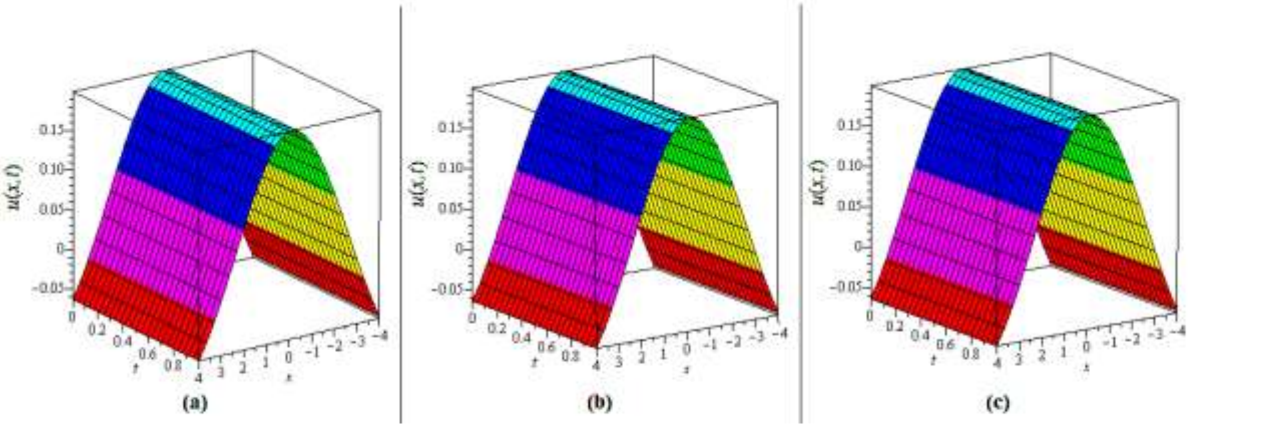

Figure 1. Plots of results of Example 1, when $-4 \leq x \leq 4, 0 \leq t \leq 1, \alpha = 2, \gamma = 1, k = 0.2$ and $a = 0.5$.

(a) Exact solution of $u(x,t)$,

(b) Approximate solution of $u(x,t)$ by HPM,

(c) Approximate solution of $u(x,t)$ by HAM with $h = -1$.

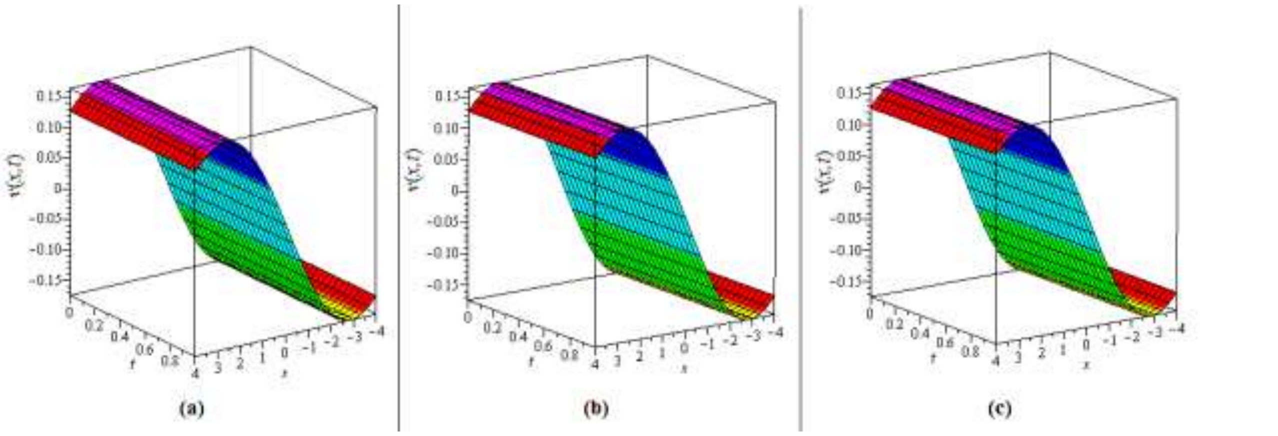

Figure 2. Plots of results for the previous Example 1, when $-4 \leq x \leq 4, 0 \leq t \leq 1, \alpha = 2, \gamma = 1, k = 0.2$ and $a = 0.5$.

(a) Exact solution of $v(x,t)$,

(b) Approximate solution of $v(x,t)$ by HPM,

(c) Approximate solution of $v(x,t)$ by HAM with $h = -1$.

Conclusion

From the comparison of HPM and HAM, through solved coupled Hirota system, it is found that the results are very close to each other especially when using the auxiliary parameter equal to -1 in the solution obtained by HAM. Also, we conclude that in the HPM the running times increase when the value of x increases from -4 to 4 while the running time of HAM is symmetrically increase with the values of $x = 0$. At most values of x the HPM is faster than HAM.

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