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Floquet Theory for Stability of Differential Algebraic Equations

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Abstract

Motivated by a great useful of some types of non autonomous differential algebraic equation systems (so called strangeness free) and its applied in different scientific fields, we present several new results for studying such systems by classical Floquet Theory, which we extended from linear periodic ordinary differential equation systems into linear periodic differential algebraic equation systems. For both systems we investigate that they have the same Floquet exponents. The relation between monodromy matrices of both systems is also presented. Classification of solution according to the nature of Floquet exponent is established. Then according to these results, we study the stability and bifurcation phenomenon of our differential algebraic equation systems.

Introduction

The dynamical behavior of physical processes is usually modeled via differential equations. But if the states of the physical system are in some ways constrained, then the mathematical model also contains algebraic equations to describe these constraints. Such systems, e.g. in mechanical multibody dynamics (Eich-Soellner , Fuhrer, 1998), electrical networks (Riaza, 2008) and chemical engineering (Kumar , Daoutidis,1999), which often cannot be modeled by standard ordinary differential equations (ODEs). These problems indeed have in common that the dynamics are algebraically constrained, for instance by tracks, Kirchhoff laws or conservation laws. A nice example can also be found in Ilchmann and Mehrmann (2005a): A mobile manipulator is modelled as a linear time-varying differential-algebraic control problem. The most general form of a differential-algebraic equation is:

$$F(t,x,\dot{x}) = 0, \qquad (1)$$

with $F: I \times D_x \times D_{\dot{x}} \longrightarrow C^m$, where $I \subseteq R$ is a compact interval , and D_x, $D_{\dot{x}} \subseteq C^m$ are open, $m,n \in N$. If $F_{\dot{x}}$ is nonsingular, then it is possible to formally solve DAEs above for $\dot{x}$in order to obtain an ordinary differential equation. However, if $F_{\dot{x}}$ is singular, this is no longer possible and the solution x has to satisfy certain algebraic constraints.The main subject in this paper is the stability in the class of differential algebraic equations (DAEs). We studied the stability of special case of the so-called strangeness free DAEs by using Floquet multipliers. Then a classification of solution of this kind of DAEs depending on the nature of Floquet multiplier is given. Also we give the conditions for the bifurcation to be occurred in the system of differential algebraic equations. These conditions included Floquet theory. Since stability phenomena gives us large understands for how the system behaves depending on states variables without knowledge the explicit of all solutions of such systems. So in this section we will study stability phenomena for some special kinds of DAEs such as linear time invariant systems (LTI), and so-called strangeness free DAEs. Which has a relation with concept of the index of the DAEs. The index of a system is the measure of the degree of singularity in this system. There are several kinds of the index phenomenon according to the nature of the system studied, such as: Kronecker index, differential index, tractability index, perturbation index, geometric index, and strangeness index. In this paper we will concert on the strangeness index. So, firstly, let us started with the linear DAEs with constant coefficient matrices which has the form

$$A\dot{x} + Bx + C = 0 \qquad (2)$$

A simultaneous transformation of A and B into Kronecker normal form makes a solution of (2). The following theorem talking about the stability of system (2).

Theorem 1.[7]: The null solution of (2) is asymptotically stable if and only if the singular values of the matrix pencil (A, B) all of have negative real part.

For the non autonomous linear DAEs, $A(t)\dot{x} + B(t)x + C(t) = 0$, the strangeness-index of DAEs is given in the following definition.

Definition 1.[4]: The minimum number$\mu$of times that all or part of the function $C(t)$in the system (2) must be differentiated in order to determine any solution $x(t)$ as a continuous function of t is the strangeness-index of the system of DAEs (2) .

Definition 2.[4]: The DAEs which has a zero strangeness-index it is called strangeness free DAEs .

In order to understand the solution behavior, we will use the derivatives of equations which has led to the concept of the strangeness index given in definition(2). This allows us to use the DAE and its derivatives to be reformulated as a strangeness free system with the same solution. That means the algebraic and differential parts are easily separated. Then for regular DAEs we may assume that the homogenous DAEs in consideration is already strangeness free. So when E is singular in the homogenous strangeness free DAE which has the form :

$$E(t)\dot{x} = A(t)x \qquad (3)$$

we can transform this DAEs into a linear ODEs, which is called essential underline ordinary differential equations (EUODE). The following theorem is essential in our work which is talking about this transformation.

Theorem 2.[8]: Consider a regular strangeness-free DAE system of the form (3) with (sufficiently smooth) coefficients E , A , let $U \in C^1(I, R^{n\times d})$ be an arbitrary orthonormal basis of the solution subspace of (3). Then there exists a matrix function $V \in C(I, R^{n\times d})$ with point wise orthonormal columns such that by the change of variables $x = Uz$ and multiplication both sides of (3) from the left by V^T, one obtain the system:

$$G\dot{z} = A_* z \ , \qquad (4)$$

where $G = V^T EU$, $A_* = V^T AU - V^T E\dot{U}$, and G is upper triangular.

For a given U, the correspondence between the solutions of (3) and those of (4) is one-one, i.e. x is a solution of (3) if and only if $z = U^T x$ is a solution of ODEs (4).

Since G is nonsingular, then (4) can be written as explicit ODE which has the form

$$\dot{z} = H(t)z, \qquad (6)$$

where $H = G^{-1}A_*$.

Now we will study the stability of linear system ODE (6) by Floquet Theory, and we will give some theorems about this ODEs. This study will be turn back the result to DAE (3) because of the correspondence of its solution with that of ODE (6) by Theorem 2.The general linear stability theory of systems with periodic coefficients is due to Floquet (1883) [5]. The fundamental solution matrix or state transition matrix (STM) $\emptyset(t)$of the system (6) is a nonsingular matrix where:

$$\dot{\emptyset}(t) = H(t)\emptyset(t), \quad \emptyset(0) = I.$$

The STM computed at the end of the principal period T, $\emptyset(t)$ is called Floquet transition matrix (FTM). The eigenvalues of (FTM), also called Floquet multipliers, determine the stability of the trivial solution for equation (6) as in the following theorem:

Theorem 3. [3]: The trivial solution of the periodic system (6) is:

(1) Asymptotically stable if and only if all Floquet multipliers lie inside the unit circle of the complex plane.

(2) Stable if and only if some of Floquet multipliers lie inside and the others on the unit circle of complex plane.

(3) Unstable if and only if at least one of Floquet multipliers lie outside the unit circle of the complex plane.

Definition 3: A monodromy matrix of the system (6) is a nonsingular matrix C associated with a fundamental matrix solution $\emptyset(t)$ of the system (6) through the relation $\emptyset(t+\omega) = \emptyset(t)C$. The eigenvalues $\rho$of a monodromy matrix are called the characteristic multipliers or Floquet multipliers, and any λ such that $\rho = e^{\lambda\omega}$ is called a characteristic exponent or Floquet exponent of the system (6).

The monodromy matrices associated with the different fundamental matrix solutions of the same linear system are similar to each other's. So the characteristic multipliers are uniquely defined, but the exponents are not. And we can always choose the Floquet exponents as the eigenvalues of B, where B is a matrix such that the monodromy matrix $C = e^{B\omega}$.

Theorem 4.(Floquet, 1883): Every fundamental matrix solution $\emptyset(t)$of the system (6) has the form

$$\emptyset(t) = P(t)\, e^{Bt}$$

where $P(t)$, B are $n \times n$ matrices, $P(t+ \omega) = P(t)$ for all t, and B is a constant matrix.

The next theorem tells us that, a study of the non autonomous linear system (6) can be reduced to a study of an autonomous linear system with constant coefficients. This theorem will be a basic in our work that's will be useful in our classification of solution of the DAEs.

Theorem 5.(Lyapunov, 1892): There exist a nonsingular periodic transformation of variables $z = P(t)\, y$, which transforms the system (6) into a system

$$\dot{y} = By \qquad (7)$$

with constant coefficients where B as defined in Theorem 4.

Since Floquet multipliers are defined in terms of fundamental matrix solution of non-autonomous linear system (6), we cannot compute them directly except that all linear independent solutions are known for a given system. In the following two theorems the Floquet multipliers of linear periodic systems can be computed without knowing their solutions.

Theorem 6.[3]: Given a homogeneous linear periodic system (6), with $H(t)$ is ω periodic and commute with antderivative ,then the monodromy matrix is given by $C = exp(\int_0^t H(\tau)\, d\tau)$, and the Floquet exponents are the eigenvalues of the matrix $B = \frac{1}{\omega}\left(\int_0^{\omega} H(\tau)\, d\tau\right)$.

Theorem 7.[3]: Consider the linear system $\dot{z} = H(t)z$. If $H(t)$ is a continuous periodic triangular matrix function, then the characteristic multipliers are given by:

$$exp\left(\int_0^{\omega} h_{11}(t)\, dt\right), exp\left(\int_0^{\omega} h_{22}(t)\, dt\right), \dots, exp\left(\int_0^{\omega} h_{nn}(t)\, dt\right),$$

and Floquet exponents are given by:

$$\frac{1}{\omega}\left(\int_0^{\omega} h_{11}(t)\, dt\right), \frac{1}{\omega}\left(\int_0^{\omega} h_{22}(t)\, dt\right), \dots, \frac{1}{\omega}\left(\int_0^{\omega} h_{nn}(t)\, dt\right).$$

According to Theorem 2 we can define the fundamental matrix solution of the DAEs (3) in the same manner as in the following:

Definition 4. $\Phi(t)$ is a fundamental solution matrix of the linear DAE system: $E(t)\dot{x} = A(t)x$ If and only if $\emptyset(t) = U^T\Phi(t)$ is a fundamental solution matrix of ODEs: $\dot{z} = H(t)z$ where $H(t) = (EU)^{-1}\left(AU - E\dot{U}\right)$, and U as defined in Theorem 2.

Therefore, Floquet theory for DAEs (3) will be formulated in the following theorem.

Theorem 8.(Floquet for DAEs): Let $E(t)\dot{x} = A(t)x$ be a linear DAE system, let $\Phi(t)$ be the fundamental matrix, then

$$\Phi(t) = Q(t)e^{Bt}$$

where $Q(t)$, B are $n \times n$ matrices, $Q(t + \omega) = Q(t)$ for all t , and B is a constant matrix.

Proof:

Since $\Phi(t)$ is a fundamental matrix of the system $E(t)\dot{x} = A(t)x$, then by definition 4, $\emptyset(t) = U^T\Phi(t)$ is a fundamental matrix of (EUODE) system $\dot{z} = H(t)z$ where $H(t)(EU)^{-1}\left(AU - E\dot{U}\right)$, and by Floquet Theorem $\emptyset(t) = P(t)e^{Bt}$, then $\Phi(t) = UP(t)e^{Bt}$ and set $\emptyset(t) = UP(t)$ which ends the proof.

Now we define the monodromy matrix for DAEs.

Definition 5: The monodromy matrix of the linear DAE system (3) is a nonsingular matrix D associated with a fundamental matrix solution $\Phi(t)$of the system (3) through relation $\Phi(t+\omega) = \Phi(t)D$.

Definition 6: Characteristic multipliers of system (3) are the eigenvalues μ of the matrix D and any λ such that $\mu = e^{\lambda\omega}$is called Floquet exponent of the system (3).

The relation between the monodromy matrix D for DAEs (3) and monodromy matrix C for (EUODE) system $\dot{z} = H(t)z$ where $H(t) = (EU)^{-1}\big(AU - E\dot{U}\big)$, is:

$$\phi(t+\omega) = \phi(t)\, C$$

$$U\,\phi(t+\omega) = U\phi(t)\, C$$

$$\Phi(t+\omega) = \Phi(t)\, C$$

But

$$\Phi(t+\omega) = \Phi(t)\, D.$$

Thus the relation between C and D is $C = D$. Then the characteristic multipliers and Floquet exponent are equals for both systems (3) and (6).

After this showing, the following theorem states the stability for the linear DAE system (3).

Theorem 9: Consider the linear DAEs (3), and assume that the matrix $H(t) = (EU)^{-1}\big(AU - E\dot{U}\big)$, is a continuous periodic lower triangular matrix function. If $\int_0^{\omega} h_{11}(t)\,dt > 0$ for some t, then the solution $x = 0$ is unstable. If all $\int_0^{\omega} h_{11}(t)\,dt < 0$ then the solution $x = 0$ is uniformly asymptotically stable. If there are some integrals equal to zero and the others are less than zero, then the system uniformly stable.

Proof:

Theorem 2 states that U is an orthonormal basis of the solutions subspace of the linear system (3) which reduced to system (6) by transformation $x = Uz$. Hence, if z is a solution of the system (6), then x is a solution of the system (3), and since that transformation preserve all properties of stability for both systems, then by Theorem 3, the proof is complete.

II. Stability of DAEs by Floquet Exponents

In this section, we introduce a different way to study the stability of the DAEs (3), this way is consist of two steps, firstly, by Theorem 2, which transforms the non autonomous strangeness free DAEs (3) into a non autonomous EUODEs (6). The second is by Lyapunov transformation which converts the non autonomous linear system (6) into linear ODEs (7) with constant coefficients. Thus, we can study the stability of the original DAE system through studying the eigenvalues of the linear ODE system with constant coefficients. That because the transformation preserve all the stability properties. For this purpose we will state the next theorem.

Theorem 10: Consider the nonautonomous linear strangeness free DAEs (3), then $x(t) = Up(t)y(t)$ (where $P(t)$ is Lyapunov transformation and U is defined in Theorem 2) is a stable solution of the system if and only if $y(t)$ is a stable solution of the linear constant ODE system (7).

Proof:

Theorem 2 states that if $x(t)$ is a solution for the linear DAE system (3), then $z(t) = U^{-1}x(t)$ is a solution for the linear ODE system (6), and by Lyapunov transformation $z(t) = P(t)y(t)$ where $y(t)$ is a solution for the constant system $\dot{y} = B\,y$, therefore, $x(t) = U\,P(t)\,y(t)$ is a solution for the linear DAE system (3). And since both of U and Lyapunov transformation preserves all properties of the stability, the proof is complete.

The fundamental theorem for linear systems tells us that the solution of the linear system (7) is given by:

$$y(t) = e^{Bt}y_0\,.$$

In the same context which followed in ODEs, to investigate the stability of solution of the linear DAEs (3) in terms of its Floquet exponents $\frac{1}{\omega}\left(\int_0^{\omega} h_{ii}(t)\,dt\right), i = 1, \dots, n,$ we will formulate some related theorems (according to the nature of these exponents). Which connect between these exponents and the eigenvalues $\lambda_i\,, i = 1, \dots, n$ of the linear ODEs (7).

For this purpose we will use Floquet Theory (Theorem 8) for linear DAEs (3).

Case 1: *Complex Distinct Floquet Exponents*

Theorem 11. If all Floquet exponents of the linear DAEs (3) are distinct complex $aj \pm ibj$, then the solution of the linear DAEs(3) is given by :

$$x(t) = UQ(t)y(t) = UQ(t)P\;diag\;[e^{a_j t}]\begin{bmatrix} cos\;b_j t & -sin\;b_j t \\ sin\;b_j t & cos\;b_j t \end{bmatrix}P^{-1}y_0, \quad y(0) = y_0$$

where $y(t)$ is the solution of the reduced system $\dot{y} = By$ with the matrix

$$P = [v_1 u_1 v_2 u_2 \dots \quad v_{n/2} u_{n/2}\,],$$

is invertible, which have columns are independent eigenvectors corresponding with the eigenvalues λ_i of the matrix B.

Proof:

We have proved that Floquet exponents for both systems (3) and (6) are equals, which are the eigenvalues of the reduced linear system with constant coefficients (7) according to (definition 3). Let the $n \times n$ real matrix B has n distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$, and corresponding complex $w_j = u_j + iv_j$ and $\bar{w}_j = u_j - iv_j\;, j = 1, \dots, \frac{n}{2},$ then from [2], the set $\{v_1 u_1 v_2 u_2 \dots \quad v_{n/2} u_{n/2}\}$ is a basis for R^n, and the solution of the initial value problem

$$\dot{y} = By, \quad y(0) = y_0,$$

is given by

$$y(t) = P\;diag\;[e^{a_j t}]\begin{bmatrix} cos\;b_j t & -sin\;b_j t \\ sin\;b_j t & cos\;b_j t \end{bmatrix}P^{-1}y_0,$$

therefore, by using Theorem 8, the solution of the linear DAE system (3) is given by:

$$x(t) = UQ(t)y(t) = UQ(t)P\ diag\ [e^{a_j t}]\begin{bmatrix} cos\ b_j t & -sin\ b_j t \\ sin\ b_j t & cos\ b_j t \end{bmatrix} P^{-1}y_0,$$

As in the same approach, we introduce the next theorem.

Case 2: Real Distinct Floquet Exponents

Theorem 12. If all Floquet exponents of the linear DAEs (3) are distinct real, then the solution of the linear DAEs (3) is given by

$$x(t) = UQ(t)y(t) = UQ(t)P\ diag\ [e^{a_j t}]\begin{bmatrix} cos\ b_j t & -sin\ b_j t \\ sin\ b_j t & cos\ b_j t \end{bmatrix} P^{-1}y_0,$$

where y(t) is the solution of the reduced system $\dot{y} = By$ with the matrix

$$P = [v_1 u_1 v_2 u_2 \dots \quad v_{n/2} u_{n/2}]$$

is invertible, which have columns are independent eigenvectors corresponding with the eigenvalues λ_i of the matrix B.

Proof:

when the eigenvalues of the linear system (7) are real distinct, then the solution is given by:[2]

$$y(t) = P\ diag\ [e^{a_j t}]P^{-1}y_0, \quad y(0) = y_0.$$

So the solution of the linear DAE system (3) is given by:

$$x(t) = UQ(t)y(t) = UQ(t)P\ diag\ [e^{a_j t}]P^{-1}y_0,$$

Sometimes, some of Floquet exponents for linear DAEs (3) are distinct real and the others are distinct complex, this case which we will deal with by the next theorem.

Theorem 13. Consider the linear DAEs (3) has k real distinct Floquet exponents and the others are distinct complex, then the solution of the system (3) is given by:

$$x(t) = UQ(t)P\ diag\ [\lambda_j \dots \ \lambda_k B_{k+1} \dots \ B_n]P^{-1}y_0,$$

when the $n \times n$ blocks$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$ for $j = k+1, \dots, n$.

Proof:

Let the real distinct Floquet exponents for the linear system (3) corresponding to the real distinct eigenvalues λ_i with eigenvector v_i, $i = 1, .., k$ of the matrix $B_{n \times n}$ of the reduced linear system (7), and the other exponents are corresponding to the eigenvalues $\lambda_j = a_j + ib_j$and $\bar{\lambda}_j = a_j - ib_j$,$j = k, \dots, n$, with corresponding eigenvector $w_j = u_j + iv_j$ and $\bar{w}_j = u_j - iv_j$ then the matrix

$$P = [v_1 \quad \dots \quad v_k\, v_{k+1}\, u_{k+1} \dots \quad v_n u_n]$$

is invertible and $P^{-1}BP = diag[\lambda_j \dots \ \lambda_k B_{k+1} \dots \ B_n]$.

and the $n \times n$ blocks

$$B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}.$$

Since the solution of the linear ODEs (7) is

$$y(t) = Pdiag[\lambda_j \dots \ \lambda_k B_{k+1} \dots \ \ B_n]P^{-1}y_0.$$

Then by Theorem 8, the solution of the linear DAE system (3) is given by

$$x(t) = UQ(t)Pdiag[\lambda_j \dots \ \lambda_k B_{k+1} \dots \ \ B_n]P^{-1}y_0.$$

This ends the proof.

Case 3: Multiple Floquet Exponents

For the repeated real Floquet exponents, we will give the following theorem.

Theorem 14. If the Floquet exponents for the linear DAEs (3) are real repeated according to their multiplicity. Then the solution of this system is given by:

$$x(t) = UQ(t)Pe^{\lambda t}P^{-1}\left[I + Nt + \dots + \frac{N^k t^k}{k!}\right]y_0,$$

where the matrix $P = [v_1 \dots \ v_n]$ is invertible, and

$$B = N + S\,P, \qquad P^{-1}SP = diag[\lambda_i].$$

The matrix $N = B - S$ is nilpotent of order $k < n,$ and S and N commute. Then, by Theorem 8, the proof is follows.

Also, when the exponents are repeated complex we have the next theorem.

Theorem 15. Consider the linear DAEs (3) with $n/2$ repeated complex exponents. Then the solution of the DAEs (3) given by

$$x(t) = UQ(t)y(t) = \ UQ(t)P\ diag\ [e^{a_j t}]\begin{bmatrix} cos\ b_j t & -sin\ b_j t \\ sin\ b_j t & cos\ b_j t \end{bmatrix} P^{-1}\left[I + Nt + \dots + \tfrac{N^k t^k}{k!}\right]y_0\ .$$

Proof:

Since there exist generalized eigenvectors $w_j = u_j + iv_j$ and $\overline{w}_j = u_j - iv_j,\ j = 1, \dots, n/2$ such that $\{u_1, v_1, \dots, u_{n/2}, v_{n/2}\}$ is a basis for R^n . For any such basis, the matrix $P = [v_1 u_1 \ \dots . v_{n/2} u_{n/2}]$ is invertible,

$$B = S + N,$$

where

$$P^{-1}SP = diag\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}.$$

The matrix N is nilpotent of order $\leq n$, and S and N commute [2].Thus according to Theorem8, the solution of the linear DAEs (3) given by

$$x(t) = UQ(t)y(t) = UQ(t)P\, diag\, [e^{a_j t}] \begin{bmatrix} cos\, b_j t & -sin\, b_j t \\ sin\, b_j t & cos\, b_j t \end{bmatrix} P^{-1} \left[I + Nt + \cdots + \frac{N^k t^k}{k!} \right] y_0.$$

Therefore, we can achieve the stability of the original linear DAE system (3) through the eigenvalues of the reduced linear ODE system (7), since in all cases that distinguished in this section, the solution of the original DAE system is

$$x(t) = UQ(t)y(t),$$

where $Q(t)$ is a Lyapunov transformation, and U is the matrix defined in the Theorem 2.For this purpose we will give the important following theorem depending on Floquet Theorem for the linear periodic system (6), with Floquet exponents are the eigenvalues $\lambda_i$for the matrix B, and Floquet multipliers are $e^{\lambda_i t}$.

Theorem 16:

(1) If all Floquet multipliers lie inside the unit complex circle, then the linear DAE system (3) is asymptotically stable.

(2) If some of the Floquet multipliers lie on the unit circle and the others inside it, then the linear DAE system (3) is stable.

(3) If at least one of the Floquet multipliers outside the unit complex circle, then the linear DAE system (3) is unstable.

Proof:

According to the Theorem 2, the linear DAE system (3) $E(t)\dot{x} = A(t)x$ reduced to the linear ODE system (6) $\dot{z} = H(t)z$ by coordinates change $x = Uz$, and by Lypunove transformation $Q(t)$, the last system is reduce to the linear ODE system (7) by change of coordinates $z = Q(t)y$. Since both two operations are preserve all properties of the stability. Then by definition 6 we have shown that characteristic multiplier and Floquet exponent are the same and the proof is follows by Theorem 3.

III. Bifurcation for Index One DAEs

We consider the semi-explicit index one DAEs

$$\dot{x} = f(x, y)$$
$$0 = g(x, y), \qquad (8)$$

where $f, g: R^2 \to R^2$ are sufficient smooth, and the parameterized differential algebraic equation for (8) is given b

$$\dot{x} = f(x, y, \mu)$$
$$0 = g(x, y, \mu), \qquad (9)$$

where $f, g: R^2 \times R \to R^2$. Let us define a function

$$F(x,y,\mu)=\begin{bmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{bmatrix}=0, \qquad (10)$$

which is equivalent to the set of critical points of (8). So when (8) has the Jacobian matrix J is lower triangular matrix (i.e $f_y = 0$ and $g_y \neq 0$), and when this Jacobian at a critical point has a zero eigenvalue, that means a bifurcation occurs at this point. This express is equivalent to say that f_x is lower triangular matrix, and it is singular at this critical point. In other words has (Floquet multipliers) which lie on the unit circle of complex or real plane.

For parameterized system DAE (8) some kinds of bifurcations which occur are explained in the next theorems. We emphasis here that the Jacobian matrix for the system (8) is a lower triangular matrix.

Theorem 17: Consider index one DAEs (8) and let the Jacobian matrix J with f_x is commutative with antiderivative, then a bifurcation occurs when one of integrations $\int_0^{\omega}(f_x)_{ii}(t)dt$ for some $t \in R$ is zero.

Proof:

Since the bifurcation requires a singular matrix f_x at the critical point, and since it is triangular matrix with entries diagonal $(f_x)_{ii}$ which lead that the eigenvalues for it are $(f_x)_{ii}(t)$, so, the bifurcation occurs when one of this eigenvalues is zero, then by Theorem 7 the proof is follows.

Theorem 18: DAE System (8) has a saddle node bifurcation at a critical point if the following conditions are satisfied

$$\int_0^{\omega}(f_x)_{ii}(t)dt = 0, f_{\mu}^0 \neq 0, f_{xx}^0 \neq 0$$

Proof:

The first condition tells us by lemma (3) that the bifurcation occurs. Now we will prove that $\mu_x^0 = 0$, and $\mu_{xx}^0 \neq 0$ which insure that the shape of bifurcation diagram is a saddle node. Second condition implies that

$$F_{\mu}^0 = 0.$$

Then by implicit function theorem equation (10) can be written as:

$$F\big(x,y,\mu(x,y)\big) = 0. \qquad (11)$$

Differentiate (11) with respect to x to get:

$$F_x + F_y\acute{y} + F_{\mu}\big(\mu_x + \mu_y\acute{y}\big) = 0.$$

Evaluating at the origin yields:

$$F_x^0 + F_{\mu}^0\mu_x^0 = 0 \Longrightarrow f_x^0 + f_{\mu}^0\mu_x^0 = 0.$$

Since singularity of the lower triangular Jacobian matrix of the index one DAEs (8) requires$f_x^0 = 0$, then we have:

$$\mu_x^0 = -f_x^0\big(f_{\mu}^0\big)^{-1} = 0.$$

Next, we differentiate (11) twice with respect to x :

$$F_{xx} + F_{xy}\acute{y} + F_{x\mu}\mu_x\acute{y} + F_y\acute{y} + [F_{xy} + F_{yy}\acute{y} + F_{y\mu}(\mu_x + \mu_y\acute{y})]\acute{y} + F_\mu[\mu_{xx} + \mu_{xy} + \mu_y\acute{y} + \acute{y}(\mu_{xy} + \mu_{yy}\acute{y})] + [\mu_x + \mu_y\acute{y}][+F_{\mu x} + +F_{\mu y}\acute{y} + F_{\mu\mu}(\mu_x + \mu_y\acute{y})] = 0.$$

Evaluating at the origin and substituting $\mu_x^0 = 0$ implies that

$$F_{xx}^0 + F_y^0\acute{y}^0 + F_\mu^0\mu_{xx}^0 + F_\mu^0\mu_y^0\acute{y}^0 = 0.$$

Which implies to

$$f_{xx}^0 + \acute{y}^0(f_y^0 + f_\mu^0\mu_y^0) + f_\mu^0\mu_{xx}^0 = 0.$$

Then by condition 3 above we can write:

$$\mu_{xx}^0 = -\left(f_\mu^0\right)^{-1}[f_{xx}^0 + \acute{y}^0(f_y^0 + f_\mu^0\mu_y^0)]$$

Finally, with both condition 2 and 3, it is certainly $\mu_{xx}^0 \neq 0$ which ends the proof.

As known in transcritical bifurcation TCB there should be two curves passing through the origin and that $F_\mu = 0$. So we cannot use the implicit function theorem, hence we write the function F given in (10) as following:

$$F(x,y,\mu) = \begin{bmatrix} xM(x,y,\mu) \\ xN(x,y,\mu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (12)$$

where the functions M and N are defined as following :

$$M(x,y,\mu) = \begin{cases} \frac{f(x,y,\mu)}{x} & if\ x \neq 0 \\ f_x & if\ x = 0. \end{cases}$$

And

$$N(x,y,\mu) = \begin{cases} \frac{g(x,y,\mu)}{x} & if\ x \neq 0 \\ g_x & if\ x = 0 \end{cases}.$$

So $x = 0$ is a surface of critical points passing through the origin from side to side of $\mu = 0$. Hence, all what we need to do is finding another surface, in $M = 0$ or $= 0$, passing through the origin from side to side of $\mu^0 = 0$, (we need to prove $\mu_x^0 \neq 0$).

Theorem 19: DAE System (8) has a transcritical bifurcation TCB at the critical point if the following conditions

$$\int_0^\omega (f_x)_{ii}(t)dt = 0, f_\mu^0 \neq 0, f_{x\mu}^0 \neq 0, f_{xx}^0 \neq 0,$$

are satisfied.

Proof:

We need to prove $\mu_x^0 \neq 0$. Since $f_{x\mu}^0 \neq 0$ which by definition of the function M means $M_\mu(x,y,\mu) \neq 0$. Then by implicit function theorem

$$M(x,y,\mu(x,y)) = 0, \qquad (15)$$

which can be differentiated with respect to x to get

$$M_x + M_y\acute{y} + M_\mu\left(\mu_x + \mu_y\acute{y}\right) = 0,$$

if we evaluated at the origin then we get

$$f_{xx}^0(x,y,\mu) + f_{x\mu}^0(x,y,\mu)\mu_x^0(x,y) = 0,$$

thus, by last conditions we get

$$\mu_x^0 \neq 0,$$

this ends the proof.

To get a pitchfork bifurcation at the critical point (the origin) we need to find two surfaces of critical points passing through the origin, one from side to side of $\mu = 0$ and the other should lay entirely in one side of $\mu = 0$, i .e ($\mu_x^0 = 0$ and $\mu_{xx}^0 \neq 0$). The following theorem talks about pitchfork bifurcation:

Theorem 20: DAE System (8) has a pitchfork bifurcation PFB at the critical point if the following conditions

$$\int_0^\omega (f_x)_{ii}(t)dt = 0, f_\mu^0 = 0, f_{x\mu}^0 \neq 0, f_{xx}^0 = 0, f_{xxx}^0 = 0,$$

are satisfied.

Proof:

We need to prove $\mu_x^0 \neq 0$ and $\mu_{xx}^0 \neq 0$. By the same way of the above proof we have $\mu_x^0 \neq 0$. It remains to prove that $\mu_{xx}^0 \neq 0$. For this, we differentiate (15) twice with respect to x:

$$\begin{aligned} M_{xx} + M_{xy}\acute{y} + M_{x\mu}\left(\mu_x + \mu_y\acute{y}\right) + M_y\acute{y} + \left[M_{xy} + M_{yy}\acute{y} + M_{y\mu}\left(\mu_x + \mu_y\acute{y}\right)\right]\acute{y} \\ + M_x\left[\mu_{xx} + \mu_{xy}\acute{y} + \mu_y\acute{y} + \left(\mu_{yx} + \mu_{yy}\acute{y}\right)\acute{y}\right] \\ + \left[M_{\mu x} + M_{\mu y}\acute{y} + M_{\mu\mu}\left(\mu_x + \mu_y\acute{y}\right)\right]\left[\mu_x + \mu_y\right]\acute{y} = 0 \end{aligned}$$

Evaluating at the origin, we get

$$\mu_{xx}^0 = -[f_{xxx}^0 + (f_{xx}^0 + f_{x\mu}^0\mu_y^0\ )\acute{y}^0](f_{x\mu}^0)^{-1}.$$

Hence by the last two conditions, we get

$$\mu_{xx}^0 \neq 0,$$

which ends the proof.

Example: Consider periodic ODE system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 + \alpha cos^2 t & 1 - \alpha \sin t \cos t\\ -1 - \alpha \sin t \cos t & -1 + \alpha sin^2 t\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

is an EUODEs for an original linear periodic DAEs(3), with

$$H(t,\alpha) = \begin{pmatrix}-1 + \alpha cos^2 t & 1 - \alpha \sin t \cos t\\ -1 - \alpha \sin t \cos t & -1 + \alpha sin^2 t\end{pmatrix}$$

and $z = (x, y)^T$ is the state variable, and α is the bifurcation parameter . This system is periodic with period $= \pi$, the state transition matrix is given by Mohler (1991) as:

$$\Phi(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix},$$

which can be factored into

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} e^{(\alpha-1)t} & 0 \\ 0 & e^{-t} \end{pmatrix} = P(t)e^{Rt},$$

where $P(t)$is the Lyapunov-Floquet transformation, and it is real and $2T$ periodic. The Floquet transition matrix is given as:

$$\Phi(\pi) = \begin{pmatrix} -e^{(\alpha-1)\pi} & 0 \\ 0 & e^{-\pi} \end{pmatrix}.$$

Since Floquet transition matrix is diagonal, hence it have eigenvalues (Floquet multipliers) $\lambda_1 = -e^{(\alpha-1)\pi}$ and $\lambda_2 = e^{-\pi}$, thus one of them λ_2 is always stable, while the other λ_1 can have a magnitude that greater than 1 if $\alpha > 1$ (lie outside of the unit circle). Therefore, this system is unstable, then by using Theorem 14 and since both systems (3) and (6) has the same monodromy matrix, then the original DAEs (3) has the same properties of stability. Also $\alpha = 1$ is the critical value of the bifurcation parameter and the critical Floquet multiplier at this point is -1 which indicates that a bifurcation occurs.

IV.Conclusion

We have presented new approach to study the stability phenomena for special case of differential algebraic equations that is linear nonautonomous strangeness free DAEs. Depending on the orthonormal matrix on the solution subspace for such system and by using an appropriate changing of variables,we can transform these systems into linear non autonomous ODE systems. The transformed system is called essential under line ordinary differential equation system EUODEs. Also we state that there is a corresponding between the solutions of the original DAE system and its reduced EUODE system. This result is very useful, since we can study the qualitative structure of the last system by Floquet Theory for the periodic linear systems. Then studying the qualitative structure of the periodic EUODEs implies to study qualitative structure of the original DAEs since the transformation preserves all properties of stability and bifurcation.

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