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Solving Two-dimensional Linear Volterra-Fredholm Integral Equations of the Second Kind by Using Series Solution Methods

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Abstract

In this paper, we focus on obtaining an approximate solution of the two types of two-dimensional linear Volterra-Fredhom integral equations of the second kind. Series solution method is reformulated and applied with different bases functions for finding an approximate solution (sometimes the exact solution) for the above two types of integral equations. This is done by computer program with the aid of the Maple code program version 13 for all the above prescribed methods. Furthermore, we proved some theoretical results on the convergence analysis of the presented methods

Introduction

The integral equations provide an important tool for modeling numerous phenomena and for solving different types of boundary value problems relating to ordinary and partial differential equations. The field of integral equations is one of the most useful mathematical tools in both pure and applied mathematics and it has wide applications in a lot of scientific problems. A very important aspect of numerical analysis is the study of the approximation to functions; usually we make such an approximation, because we wish to carry out some numerical calculation or analytical operations involving these functions such as differentiation and integration. One of the most important tools of the approximation theory is the power series expansion, by which we can replace certain complicated functions by an approximated or equal simple one [14].

Many phenomena in physics and engineering fields give rise to two-dimensional integral equations. Many integral equations are usually difficult to solve analytically. In many cases, it's required to obtain the approximate solutions. Numerical methods for solving integral equations have always been important in applied sciences. With the advent of computers, the use of numerical methods has spread, and more importantly, people are now able to tackle those problems which are fundamental to our understanding of scientific phenomenon; but that were much more difficult to study in previous works. Integral equations that involve functions of two or more independent variables occur frequently in the study of many problems in partial differential equations which arise from various dynamic models. As we know, much work has been done on developing and analyzing numerical methods for solving one-dimensional integral equations of the second kind, but in two-dimensional cases, a rather small amount of work has been done.

The two-dimensional Volterra-Fredholm integral equations are solved by the following authors: In 1986, the time collocation method is used for solving mixed Volterra-Fredholm integral equations and Volterra-Fredholm integral equations by [12] and [8] respectively. A techniques based on the Adomian decomposition method are used for solving mixed Volterra-Fredholm integral equations by [9] and [15]. Homotopy Perturbation Method used for solving two-dimensional Volterra-Fredholm Integral equations by [4], while two-dimensional Legendre Wavelets Method applied to solve mixed Volterra-Fredholm integral equations by [3]. For more details on the numerical solutions of two-dimensional Volterra-Fredholm integral equations, we refer the reader to the following works [7, 10, 2, 17, 1, 18]. Power series expansion and its properties are used to find an approximated solution of different types of integral equations and differential equations [5, 13, 16].

In this paper, we focus on obtaining an approximate solution of the following two-dimensional linear Volterra-Fredhom integral equations of the second kind:

- Two-dimensional linear mixed Volterra-Fredhom integral equations of the second kind (MVFIEK2's).

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z)u(y,z)dydz, \quad (1)$$

- Two-dimensional linear Separate Volterra-Fredhom integral equations of the second kind (SVFIEK2's).

$$u(x,t) = f(x,t) + \lambda_1 \int_0^t \int_0^x k_1(x,t,y,z)u(y,z)dydz + \lambda_2 \int_0^1 \int_0^1 k_2(x,t,y,z)u(y,z)dydz, \quad (2)$$

where $u(x,t)$ is considered an unknown function to be determined, the functions $f(x,t)$, $k(x,t,y,z)$ are analytic on $\mathcal{Q} = [0,1) \times [0,1)$, the functions $k_1(x,t,y,z)$ and $k_2(x,t,y,z)$ are given functions defined on $\mathcal{G} = \{(x,t): 0 \le y \le x < 1, 0 \le z \le t < 1\}$ by using Series solution with power basis function and orthogonal bases functions as follows:

A. Power Series Basis Function (PSBF)

Let $u(x)$ be an analytic function, then the Taylor series of $u(x)$ at any point b in its domain is given by

$$u(x) = \sum_{i=0}^{\infty} \frac{u^{(i)}(b)}{i!}(x-b)^i, \quad (3)$$

which converges to $\mathbf{u}(\mathbf{x})$. For simplicity, in this paper we can use Maclaurin series for $\mathbf{u}(\mathbf{x})$ which is written as follows:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \quad (4)$$

In this section we will present a useful method that stems mainly from the Taylor series for analytic functions, for solving linear Volterra-Fredholm integral equations. We will assume that a solution $u(x)$ of the mixed Volterra-Fredholm integral equation

$$u(x) = f(x) + \int_0^x \int_a^b K(s,t)\, u(t)dtds, \quad (5)$$

is analytic, and therefore possesses a Taylor series of the form given in (4), where the coefficients a_n will be determined in an algebraic manner. In this method, we usually substitute the Taylor series (4) into both sides of (5) to obtain

$$\sum_{n=0}^{\infty} a_n x^n = Taylor(f(x)) + \int_0^x \int_a^b K(s,t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dtds, \quad (6)$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots =$$
$$Taylor\big(f(x)\big) + \int_0^x \int_a^b K(s,t)(a_0 + a_1 t + a_2 t^2 + \cdots) dtds, \qquad (7)$$

where $Taylor(f(x))$ is the Taylor series for $f(x)$.

The mixed Volterra-Fredholm integral equation (MVFIEK2's) (5) will be converted to a traditional integral in (7) where instead of integrating the unknown function $u(x)$, terms of the form t^n for $n = 0, 1, \ldots$ will be integrated. If $f(x)$ includes one or for of the following functions; trigonometric functions, exponential functions, etc., then Taylor series expansions for functions involved in $f(x)$ must be used.

After finding the integrals in (7), and collecting the coefficients of like powers of x, then, we equate the coefficients of like powers of x into both sides of the resulting equation in order to determine a system of linear algebraic equations in a_j for $j = 0, 1, \ldots$. Solving the resulting system of linear equations for finding the values of the coefficients a_j for $j = 0, 1, \ldots$. Approximate solution follows immediately after substituting the values of $a_j^{'}s$ in (4). For more details about PSBF, see [16].

A.1 Solving MVFIEK2's by using PSBF

In this section, we reformulate PSBF for solving (1) as follows:
First, we substitute

$$u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i t^j, \qquad (8)$$

into both side of (1) to get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i t^j = Taylor\big(f(x,t)\big) + \lambda \int_0^t \int_a^b K(x,t,y,z) \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} y^i z^j \right) dydz, \qquad (9)$$

or, for simplicity we use

$$a_{0,0} + a_{0,1} t + a_{0,2} t^2 + \cdots + a_{1,0} x + a_{2,0} x^2 + \cdots$$
$$= Taylor\big(f(x,t)\big) + \int_0^t \int_a^b k(x,t,y,z)\,(a_{0,0} + a_{0,1} z + a_{0,2} z^2$$
$$+ \cdots + a_{1,0} y + a_{2,0} y^2 + \cdots) dydz, \qquad (10)$$

where $Taylor(f(x,t))$ is the Taylor series for $f(x,t)$. If $f(x,t)$ includes one or more of the following functions such; trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x,t)$ must be used. After finding the integrals in (10), and collecting the coefficients of like powers of x, then, we equate the coefficients of like powers of x into both sides of the resulting equations in order to determine a system of algebraic equations in $a_{i,j}$ for $i,j = 0, 1, \ldots$. Solving the resulting system of linear equations for finding the values of the coefficients $a_{i,j}$ for $i,j = 0, 1, \ldots$. Approximate solutions of (1) follows immediately after substituting the values of $a_{i,j}$'s in (8).

A.2 Solving SVFIEK2's by using PSBF

In this section, we reformulate PSBF for solving (2) as follows:
First, we substitute (8) into both side of (2) to get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i t^j = Taylor\big(f(x,t)\big)$$
$$+\lambda_1 \int_0^1 \int_0^1 k_1(x,t,y,z) (\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} y^i z^j)\, dydz \;\; + \lambda_2 \int_0^t \int_0^x k_2(x,t,y,z) (\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} y^i z^j)\, dydz, \qquad (11)$$

or, for simplicity we use

$$a_{0,0}+a_{0,1}t+a_{0,2}t^2+\cdots+a_{1,0}x+a_{2,0}x^2+\cdots=Taylor\big(f(x,t)\big)$$
$$+\lambda_1\int_0^1\int_0^1 k_1(x,t,y,z)(a_{0,0}+a_{0,1}z+a_{0,2}z^2+\cdots+a_{1,0}y+a_{2,0}y^2+\cdots)dydz$$
$$+\lambda_2\int_0^t\int_0^x k_2(x,t,y,z)\,(a_{0,0}+a_{0,1}z+a_{0,2}z^2\;+\cdots+a_{1,0}y+a_{2,0}y^2+\cdots)dydz. \quad (12)$$

If $f(x,t)$ includes one or more of the following functions; trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x,t)$ must be used. After finding the integrals in (12), and collecting the coefficients of like powers of x, then, we equate the coefficients of like powers of x into both sides of the resulting equations in order to determine a system of algebraic equations in $a_{i,j}$ for $i,j=0,1,\ldots$. Solving the resulting system of linear equations for finding the values of the coefficients $a_{i,j}$ for $i,j=0,1,\ldots$. Approximate solutions of (2) follows immediately after substituting the values of $a_{i,j}$'s in (8).

B. Orthogonal Basis Functions (OBF)

In this section, we describe a power series solution with three types of orthogonal polynomials (Chebyshev for the first kind, Legendre and Hermite polynomials) as a basis function for solving MVFIEKS'2 and SVFIEKS'2.

B.1 Solving MVFIEK2's by using OBF

First, for solving MVFIEK2's we would like to describe series solution method by using Chebyshev polynomial of the first kind as a basis function, and for the other two bases function the methods are similar. The approximate solution of (1) proposed in the form:

$$u_{n,m}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}T_i(x)T_j(t), \quad (13)$$

where $T_i(x)$ and $T_j(t), i=0,1,\ldots,n; j=0,1,\ldots,m; n,m\in\mathbb{N}$ are the Chebyshev polynomials and the control points $c_{i,j}$ are undermined constants coefficients.

To obtain the solution, substitute (13) in (1) to get:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}T_i(x)T_j(t)=\;Taylor\big(f(x,t)\big)+\lambda\int_0^t\int_a^b k(x,t,y,z)\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}\,T_i(y)T_j(z)dydz, \quad (14)$$

or

$$c_{0,0}T_0(x)T_0(t)+c_{0,1}T_0(x)T_1(t)+\cdots+c_{n,m}T_n(x)T_m(t)=Taylor\big(f(x,t)\big)+$$
$$\lambda\int_0^t\int_a^b k(x,t,y,z)(c_{0,0}T_0(y)T_0(z)+c_{0,1}T_0(y)T_1(z)+\cdots+c_{n,m}T_n(y)T_m(z))dydz. \quad (15)$$

Now we have to find all integration in Equation (15). In order to determine $c_{i,j}, i=0,1,\ldots,n; j=0,1,\ldots,m; n,m\in\mathbb{N}$ we need $(n+1)\times(m+1)$ linear system of equations; for this reason we equate the coefficients of like powers of x and t or together after integrating Equation (15).

Finally solve the resulting $(n+1)\times(m+1)$ linear system of equations by using Gauss elimination to obtain the values $c_{i,j}, i=0,1,\ldots,n; j=0,1,\ldots,m; n,m\in\mathbb{N}$. Substitute these values in (13) to obtain approximate solution of (1).

B.2 Solving SVFIEK2's by using OBF

First, for solving SVFIEK2's we describe series solution method by using Chebyshev polynomial of the first kind as a basis function, and for the other two bases function the method are similar.

To obtain the approximate solution, substitute (13) into (2), yields:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}T_i(x)T_j(t) = Taylor\big(f(x,t)\big) + \lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}T_i(y)T_j(z)\,dydz$$
$$+\lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)\sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}T_i(y)T_j(z)\,dydz, \tag{16}$$

or

$$c_{0,0}T_0(x)T_0(t) + c_{0,1}T_0(x)T_1(t) + \cdots + c_{n,m}T_n(x)T_m(t) = Taylor\big(f(x,t)\big) +$$
$$\lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)\,(c_{0,0}T_0(y)T_0(z) + c_{0,1}T_0(y)T_1(z) + \cdots + c_{n,m}T_n(y)T_m(z))dydz$$
$$+\lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)(c_{0,0}T_0(y)T_0(z) + c_{0,1}T_0(y)T_1(z) + \cdots + c_{n,m}T_n(y)T_m(z))dydz. \tag{17}$$

Now we have to find all integration in Equation (17). In order to determine $c_{i,j}, i = 0,1,\dots,n; j = 0,1,\dots,m; n,m \in \mathbb{N}$ we need $(n+1)\times(m+1)$ linear system of equations; for this reason we equate the coefficients of like powers of x and t or together after integrating Equation (17).

Finally solve the resulting $(n+1)\times(m+1)$ linear system of equations by using Gauss elimination to obtain the values $c_{i,j}, i = 0,1,\dots,n; j = 0,1,\dots,m; n,m \in \mathbb{N}$. Substitute these values in (13) to obtain approximate solution of (2).

C. Convergent Analysis

In this section, we study the uniqueness and convergence of PSBF and OBF for solving Equations (1) and (2).

C.1 Convergent Analysis of PSBF Used for Solving MVFIEK2's

Let $(C([a,b]\times[0,c_1]), \|.\|)$ be the space of all continuous functions on the interval $[0,c_1]\times[a,b]$ with the norm

$$\|u\|_\infty = \underbrace{max}_{\substack{x\in[a,b]\\ t\in[0,c_1]}} |u(x,t)|.$$

We assume that $u(x,t) \neq 0$, $|k(x,t,y,z)| \leq M$, (M positive real number) $\forall (x,t) \in [a,b]\times[0,c_1]$ and $\mathfrak{P} = \{(x,t,y,z): 0 \leq z \leq t \leq c_1, a \leq y \leq x \leq b\}$.

With these conditions, we present the following theorem:

Theorem 1

Let $u(x,t)$ be an exact solution of the Equation (1) and $u_{N,N}(x,t)$ be the approximate solution of (1) where

$$u_{N,N}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{i.j}x^i t^j.$$

Then, the solution of MVFIEK2's by using power series based function is unique and convergent if $0 < \alpha < 1$.

Proof

First, we proof the uniqueness. Let $u_{N,N}$ and $u'_{N,N}$ be two different approximate solutions for (1). By the presented method in Subsection A.1, we will have

$$\begin{aligned}
&\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \\
&= \left|Taylor\big(f(x,t)\big) + \lambda \int_0^t \int_a^b k(x,t,y,z) u_{N,N}(y,z)\, dydz - Taylor(f(t,x)) \right.\\
&\left. - \lambda \int_0^t \int_a^b k(x,t,y,z) u'_{N,N}(y,z)\, dydz \right| \\
&= \left|\lambda \int_0^t \int_a^b k(x,t,y,z)\, u_{N,N}(y,z)\, dydz - \lambda \int_0^t \int_a^b k(x,t,y,z) u'_{N,N}(y,z)\, dydz\right| \\
&= \left|\lambda \int_0^t \int_a^b k(x,t,y,z)(u_{N,N}(y,z) - u'_{N,N}(y,z))\, dydz\right| \\
&\le |\lambda| \int_0^t \int_a^b |k(x,t,y,z)| \left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right| dydz \\
&\le |\lambda| M \int_0^t \int_a^b \left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right| dydz \\
&\le |\lambda| M \mathfrak{P} \left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| = \alpha \left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right|,
\end{aligned}$$

where $\alpha = |\lambda| M \mathfrak{P}$.

Then $\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \le \alpha \left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right|$,

from which we get $(1-\alpha)\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \le 0$.

Since $0 < \alpha < 1$, then $\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| = 0$, and this implies that $u_{N,N} = u'_{N,N}$. Hence, the uniqueness proof is complete.

Now, we proof the convergence. From the definitions of the norms, we have

$$\begin{aligned}
&\left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty = \underbrace{max}_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \left|u(x,t) - u_{N,N}(x,t)\right| \\
&= \underbrace{max}_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \left|f(x,t) + \lambda \int_0^t \int_a^b k(x,t,y,z) u(y,z)\, dydz - Taylor(f(x,t)) \right.\\
&\left. - \lambda \int_0^t \int_a^b (k(x,t,y,z) u_{N,N}(y,z)) dydz \right| \\
&\le \underbrace{max}_{\substack{x\in[a,b]\\ t\in[0,c_1]}} |f(x,t) - Taylor(f(x,t))| + \lambda \underbrace{max}_{\substack{x\in[a,b]\\ t\in[0,c_1]}} \int_0^t \int_a^b |k(x,t,y,z)|\, \left|u(y,z) - u_{N,N}(y,z)\right| dydz \\
&\le \|f(x,t) - Taylor(f(x,t))\| + \lambda M \mathfrak{P} \left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty \quad \text{(see[11])} \\
&\le \varepsilon + \lambda M \mathfrak{P} \left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty .
\end{aligned}$$

Since $\varepsilon \to 0$, hence we have $\left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty \le \alpha \left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty$,

where $\alpha = \lambda M \mathfrak{P}$, or $(1-\alpha)\left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty \le 0$.

Then, if $0 < \alpha < 1$ and $N \to \infty$ we have $\lim_{N\to\infty} \left\|u(x,t) - u_{N,N}(x,t)\right\|_\infty = 0$.

This shows that the method is converge. ■

C.2 Convergent Analysis of PSBF Used for Solving SVFIEK2's

Let $\big(C([0,1) \times [0,1)), \|.\|\big)$ be the space of all continuous functions on interval $[0,1) \times [0,1)$with the norm

$$\|u\|_\infty = \underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}} |u(x,t)|.$$

We assume that $u(x,t) \neq 0$, $|k_1(x,t,y,z)| \leq M_1$ and $|k_2(x,t,y,z)| \leq M_2$ for all $(x,t) \in [0,1) \times [0,1)$ and $\mathcal{G} = \{(x,t): 0 \leq y \leq x < 1, 0 \leq z \leq t < 1\}$ M_1 and M_2 where are two positive real numbers. With these conditions, we present the following theorem:

Theorem 2

Let $u(x,t)$ be an exact solution of the Equation (2) and $u_{N,N}(x,t)$ be the approximate solution of the Equation (2) where

$$u_{N,N}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{i.j} x^i t^j.$$

The solution of the SVFIEK2's by using power series based function is unique and convergent if $0 < \beta < 1$.

Proof

First, we proof uniqueness. Let $u_{N,N}$ and $u'_{N,N}$ be two different approximate solutions for (2). By the presented method in the Subsection A.2, we have

$$\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \left|Taylor(f(x,t)) + \lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)u_{N,N}(y,z)dydz \right.$$
$$+ \lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)u_{N,N}(y,z)dydz - Taylor(f(x,t))$$
$$\left. - \lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)u'_{N,N}(y,z)dydz - \lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)u'_{N,N}(y,z)dydz\right|$$
$$= \left|\lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)u_{N,N}(y,z)dydz + \lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)u_{N,N}(y,z)dydz\right.$$
$$\left. - \lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)u'_{N,N}(y,z)dydz - \lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)u'_{N,N}(y,z)dydz\right|$$
$$= \left|\lambda_1 \int_0^1\int_0^1 k_1(x,t,y,z)\left(u_{N,N}(y,z) - u'_{N,N}(y,z)\right)dydz + \lambda_2 \int_0^t\int_0^x k_2(x,t,y,z)\left(u_{N,N}(y,z) - u'_{N,N}(y,z)\right)dydz\right|$$
$$\leq |\lambda_1| \int_0^1\int_0^1 |k_1(x,t,y,z)|\left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right|dydz$$
$$+|\lambda_2| \int_0^t\int_0^x |k_2(x,t,y,z)|\left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right|dydz$$
$$\leq |\lambda_1|M_1 \int_0^1\int_0^1 \left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right|dydz + |\lambda_2|M_2 \int_0^t\int_0^x \left|u_{N,N}(y,z) - u'_{N,N}(y,z)\right|dydz$$
$$\leq |\lambda_1|M_1\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| + |\lambda_2|M_2\mathcal{G}\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right|$$
$$= \beta\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right|,$$

where $\beta = |\lambda_1|M_1 + |\lambda_2|M_2\mathcal{G}$. Then

$$\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \leq \beta\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right|,$$

From which we get $(1-\beta)\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| \leq 0$.
Since $0 < \beta < 1$ then $\left|u_{N,N}(x,t) - u'_{N,N}(x,t)\right| = 0$.
This implies $u_{N,N} = u'_{N,N}$. Hence the uniqueness proof is complete.

Now, we proof convergence. From the definition of norm we have

$$\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}=\underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}}\left|u(x,t)-u_{N,N}(x,t)\right|$$

$$=\underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}}\left|f(x,t)+\lambda_1\int_0^1\int_0^1 k_1(x,t,y,z)u(y,z)dydz+\lambda_2\int_0^t\int_0^x k_2(x,t,y,z)u(y,z)dydz-Taylor(f(x,t))\right.$$

$$\left.-\lambda_1\int_0^1\int_0^1 k_1(x,t,y,z)(u_{N,N}(y,z))dydz-\lambda_2\int_0^t\int_0^x k_2(x,t,y,z)(u_{N,N}(y,z))dydz\right|$$

$$\leq \underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}}\left|f(x,t)-Taylor(f(x,t))\right|$$

$$+|\lambda_1|\underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}}\int_0^1\int_0^1|k_1(x,t,y,z)|\left|u(y,z)-u_{N,N}(y,z)\right|dydz$$

$$+|\lambda_2|\underbrace{max}_{\substack{x\in[0,1)\\ t\in[0,1)}}\int_0^t\int_0^x|k_2(x,t,y,z)|\left|u(y,z)-u_{N,N}(y,z)\right|dydz$$

$$\leq\|f(x,t)-Taylor(f(x,t))\|+|\lambda_1|M_1\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}$$

$$+|\lambda_2|M_2\mathcal{G}\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}\qquad (see[11])$$

$$\leq\varepsilon+|\lambda_1|M_1\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}+|\lambda_2|M_2\mathcal{G}\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}.$$

Since $\varepsilon\to 0$, hence we have

$$\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}\leq\beta\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty},$$

where $\beta=|\lambda_1|M_1+|\lambda_2|M_2\mathcal{G}$ or, $(1-\beta)\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}\leq 0$.

Then, if $0<\beta<1$ and $N\to\infty$ we have

$$\lim_{N\to\infty}\left\|u(x,t)-u_{N,N}(x,t)\right\|_{\infty}=0.$$

This completes the convergence proof. ■

C.3 Convergent Analysis of OBF Used for Solving MVFIEK2's

Under the same conditions given in the Subsection B.1, we prove the following theorem:

Theorem 3

Let $u(x,t)$ be an exact solution of the Equation (1) and $u_{n,m}(x,t)$ be the approximate solution of the Equation (1) where

$$u_{n,m}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i.j}\,Q_i(x)Q_j(t),$$

and $Q_i(x)$, $Q_j(t)$ for $0\leq i\leq n, 0\leq j\leq m;\ n,n\in\mathbb{N}$ are the orthogonal basis functions (one of the following orthogonal polynomials: Chebyshev of the first kind, Legendre and Hermite). Then the solution of the MVFIEK2's by using orthogonal basis function is unique and convergent if $0<\alpha<1$.

Proof

The proof of this Theorem is similar to the proof of Theorem 1, for this reason we omit it. ■

C.4 Convergent Analysis of OBF Used for Solving SVFIEK2's

Under the same conditions given in the Subsection B.2, we prove the following theorem:

Theorem 4

Let $u(x,t)$ be an exact solution of the Equation (2) and $u_{n,m}(x,t)$ be the approximate solution of the Equation (2) where

$$u_{n,m}(x,t)=\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i.j}Q_i(x)Q_j(t),$$

and $Q_i(x)$, $Q_j(t)$ for $0\le i\le n, 0\le j\le m; n,m\in\mathbb{N}$ are the orthogonal basis function (one of the following orthogonal polynomials: Chebyshev of the first kind, Legendre and Hermite). Then the solution of the SVFIEK2's by using orthogonal basis function is unique and convergent if $0<\beta<1$.

Proof

The proof of this Theorem is similar to the proof of Theorem 2 for this reason we omit it. ■

Remark:

All those four theorems are talked about non-homogeneous integral equations but if we have the homogeneous integral equations the proof are almost the same.

Numerical Examples

To show the validity of the methods which will be presented in this paper depending on the least square errors (L.S.E) and running times R.T, the following four test examples are solved numerically:

Example 1 [6]

The exact solution of this integral equation

$$u(x,t)=x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5+\int_0^t\int_0^1(x\,ty^2\,z^2)\,u(y,z)dydz\,.$$

is $u(x,t)=x^2+xt$.

Solution:

(i) Using PSBF:

Let $N=3$ and substitute

$$u(x,t)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{ij}\,x^i t^j, \tag{18}$$

into both sides of the integral equation in the Example 1, we get

$$a_{0,0}+a_{0,1}t+a_{0,2}t^2+a_{0,3}t^3+a_{1,0}x+a_{1,1}xt+a_{1,2}xt^2+a_{1,3}xt^3+a_{2,0}x^2+a_{2,1}x^2t+a_{2,2}x^2t^2$$
$$+a_{2,3}x^2t^3+a_{3,0}x^3+a_{3,1}x^3t+a_{3,2}x^3t^2+a_{3,3}x^3t^3=x^2+xt-\frac{1}{15}xt^4-\frac{1}{16}xt^5$$
$$+\int_0^t\int_0^1(x\,ty^2\,z^2)\,(a_{0,0}+a_{0,1}z+a_{0,2}z^2+a_{0,3}z^3+a_{1,0}y+a_{1,1}yz+a_{1,2}yz^2+a_{1,3}yz^3+a_{2,0}y^2$$
$$+a_{2,1}y^2z+a_{2,2}y^2z^2+a_{2,3}y^2z^3+a_{3,0}y^3+a_{3,1}y^3z+a_{3,2}y^3z^2+a_{3,3}y^3z^3)dydz\,.$$

Now evaluating the integral at the right side of the above integral equation, collecting the coefficients of like powers of x,t and xt and equating the coefficients of like powers of x,t and xt in both sides, we obtain

$$a_{0,0}=0, a_{0,1}=0,\ a_{0,2}=0,\ a_{0,3}=0,\ a_{1,0}=0,\ a_{1,1}=1, a_{1,2}=0,$$
$$a_{1,3}=0, a_{2,0}=1, a_{2,1}=0,\ a_{2,2}=0, a_{2,3}=0,\ a_{3,0}=0,\ a_{3,1}=0,$$
$$a_{3,2}=0, a_{3,3}=0.$$

Finally Substitute these coefficients into (18), we obtain

$u_{3,3}(x,t)=x^2+xt$ which is the exact solution.

Note:

The errors equal to zero after $N = 2$.

(ii) Using OBF: The solution with $n = m = 3$ for all three types of the bases functions are:

$u_{3,3}(x,t) = x^2 + xt.$

Note:

The errors equal to zero after $n = m = 2$.

Example 2 [4]

The exact solution of this integral equation

$$u(x,t) = t^2 e^x - \frac{2}{3}\, x^2 t^3 + \int_0^t \int_{-1}^{1} (x^2\, e^{-y})\, u(y,z) dy dz.$$

is $u(x,t) = t^2 e^x$.

Solution:

(i) Using PSBF: The solution with $N = 3$ is:

$$u_{3,3}(x,t) = t^2 + t^2 x + \tfrac{1}{2} t^2 x^2 + \tfrac{1}{6} t^2 x^3 - 0.0069440287 t^3 x^2.$$

(ii) Using OBF: The solution with $n = m = 3$ are:

(a) By Chebyshev polynomial:

$$\begin{aligned} u_{3,3}(x,t) = {} & \frac{1}{48} e^{-1}(11e^2 - 49 - 12e)t + \frac{5}{4} t^2 + \frac{1}{144} e^{-1}(11e^2 - 49 - 12e)(4t^3 - 3t) + \frac{1}{2} x \\ & + \frac{9}{16} x(2t^2 - 1) + \frac{1}{4} x^2 - \frac{1}{8} + \frac{1}{48} e^{-1}(11e^2 - 49 - 12e)(2x^2 - 1)t \\ & + \frac{1}{8}(2x^2 - 1)(2t^2 - 1) + \frac{1}{144} e^{-1}(11e^2 - 49 - 12e)(2x^2 - 1)(4t^3 - 3t) + \frac{1}{12} x^3 \\ & + \frac{1}{48}(4x^3 - 3x)(2t^2 - 1). \end{aligned}$$

(b) By Legendre polynomial:

$$\begin{aligned} u_{3,3}(x,t) = {} & \frac{1}{90} t e^{-1}(-49 + 11e^2 - 12e) + \frac{7}{6} t^2 + \frac{1}{135} e^{-1}(-49 + 11e^2 - 12e)\left(\frac{5}{2} t^3 - \frac{3}{2} t\right) + \frac{1}{3} x \\ & + \frac{11}{15} x\left(\frac{3}{2} t^2 - \frac{1}{2}\right) + \frac{1}{6} x^2 - \frac{1}{18} + \frac{1}{45} e^{-1}(-49 + 11e^2 - 12e)\left(\frac{3}{2} x^2 - \frac{1}{2}\right) t \\ & + \frac{2}{9}\left(\frac{3}{2} x^2 - \frac{1}{2}\right)\left(\frac{3}{2} t^2 - \frac{1}{2}\right) + \frac{2}{135} e^{-1}\left(\frac{3}{2} x^2 - \frac{1}{2}\right)\left(\frac{5}{2} t^3 - \frac{3}{2} t\right) + \frac{1}{18} x^3 \\ & + \frac{2}{45}\left(\frac{5}{2} x^3 - \frac{3}{2} x\right)\left(\frac{3}{2} t^2 - \frac{1}{2}\right). \end{aligned}$$

(c) By Hermit polynomial:

$$\begin{aligned} u_{3,3} = {} & \frac{1}{24} e^{-1}(11e^2 - 49 - 12e)t + \frac{5}{4} t^2 + \frac{1}{288} e^{-1}(11e^2 - 49 - 12e)(8t^3 - 12t) + \frac{1}{2} x \\ & + \frac{5}{16} x(4t^2 - 2) + \frac{1}{4} x^2 - \frac{1}{8} + \frac{1}{48} e^{-1}(11e^2 - 49 - 12e)(4x^2 - 2)t \\ & + \frac{1}{32}(4x^2 - 2)(4t^2 - 2) + \frac{1}{576} e^{-1}(11e^2 - 49 - 12e)(4x^2 - 2)(8t^3 - 12t) + \frac{1}{12} x^3 \\ & + \frac{1}{192}(8x^3 - 12x)(4t^2 - 2). \end{aligned}$$

Example 3

The exact solution of this integral equation

$$u(x,t) = x^2 + \frac{7}{4}xt - \frac{2}{9}t^3x^3 - \frac{1}{2}\left(\frac{1}{4}x^4 + x^3t\right)t^2 - \frac{1}{3}x^4t^2 + \int_0^1\int_0^1 (xtyz^2)\, u(y,z)dydz + \int_0^t\int_0^x (xt+yz)\, u(y,z)dydz.$$

is $u(x,t) = x^2 + 2xt$.

Solution:

(i) Using PSBF: The solution with $N = 3$ is:

$$u_{3,3}(x,t) = x^2 + 2xt.$$

Note:

The errors equal to zero after $N = 2$.

(ii) Using OBF: The solution with $n = m = 3$ for all three types of the polynomials are:

$$u_{3,3}(x,t) = x^2 + 2xt.$$

Note:

The errors equal to zero after $n = m = 2$.

Example 4

The exact solution of this integral equation

$$\mathrm{u(x,t) = te^x - \frac{5}{6}t + \frac{1}{3}te - \frac{1}{2}(-1+e^x)t^2 + \int_0^1\int_0^1 (t(y-z)\, u(y,z)dydz + \int_0^t\int_0^x u(y,z)dydz.}$$

is $\mathrm{u(x,t) = te^x}$.

Solution:

(i) Using PSBF: The solution with $N = 3$ is:

$$u_{3,3}(x,t) = 0.9992201702t + tx + \frac{1}{2}tx^2 + \frac{1}{3}tx^3 - 0.00006498581t^3x^2.$$

(ii) Using OBF: The solution with $n = m = 3$ are:

(a) By Chebyshev polynomial:

$$u_{3,3}(x,t) = \left(\frac{3287}{8376} + \frac{110}{349}e\right)t + \left(-\frac{653}{75384} + \frac{10}{3141}e\right)(t^3 - 3t) + \left(-\frac{653}{3141} + \frac{80}{1047}e\right)x + \frac{9}{8}xt + \left(-\frac{653}{3141} + \frac{80}{1047}e\right)(2t^2-1)x + \left(\frac{5629}{25128} + \frac{10}{1047}e\right)(2x^2-1)t + \left(-\frac{653}{75384} + \frac{10}{3141}e\right)(2x^2-1)(4t^3-3t) + \frac{1}{24}(4x^3-3x)t.$$

(b) By Legendre polynomial:

$$u_{3,3}(x,t) = \left(\frac{30269}{94230} + \frac{976}{3141}e\right)t + \left(-\frac{1306}{141345} + \frac{32}{9423}e\right)\left(\frac{5}{2}t^3 - \frac{3}{2}t\right) + \left(-\frac{1306}{9423} + \frac{160}{3141}e\right)x + \frac{11}{10}tx + \left(-\frac{2612}{9423} + \frac{320}{3141}e\right)x\left(\frac{3}{2}t^2 - \frac{1}{2}\right) + \left(\frac{14399}{47115} + \frac{32}{3141}e\right)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)t + \left(-\frac{2612}{141345} + \frac{64}{9423}e\right)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\left(\frac{5}{2}t^3 - \frac{3}{2}t\right) + \frac{1}{15}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)t.$$

(c) By Hermit polynomial:

$$u_{3,3}(x,t)=\left(\frac{1151}{6282}+\frac{170}{1047}e\right)2\,t+\left(-\frac{653}{150768}+\frac{5}{3141}e\right)(8\,t^3-12t)+\left(-\frac{653}{6282}+\frac{40}{1047}e\right)2x$$
$$+\frac{5}{4}\,tx+\left(-\frac{653}{12564}+\frac{20}{1047}e\right)2\,x(4t^2-2)+\left(\frac{311}{6282}+\frac{5}{1047}e\right)(4x^2-2)2t$$
$$+\left(-\frac{653}{301536}+\frac{5}{6282}e\right)(4x^2-2)(8t^3-12t)+\frac{1}{48}(8x^3-12x)t.$$

Note:

1. **CHOBF** denotes Chebyshev polynomial for the First Kind as an Orthogonal Basis Function.
2. **LEOBF** denotes Legendre polynomial as an Orthogonal Basis Function.
3. **HEOBF** denotes Hermite polynomial as an Orthogonal Basis Function.

Table (1a) The results for the approximate solution $\boldsymbol{u(x,t)}$ of Example 1, by using **PSBF**, **CHOBF**, **LEOBF** and **HEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.02	0.02	0.02	0.02
(0.2,0.2)	0.08	0.08	0.08	0.08
(0.3,0.3)	0.18	0.18	0.18	0.18
(0.4,0.4)	0.32	0.32	0.32	0.32
(0.5,0.5)	0.50	0.50	0.50	0.50
(0.6,0.6)	0.72	0.72	0.72	0.72
(0.7,0.7)	0.98	0.98	0.98	0.98
(0.8,0.8)	1.28	1.28	1.28	1.28
(0.9,0.9)	1.62	1.62	1.62	1.62
(1,1)	2	2	2	2
$\boldsymbol{L.S.E.}$ PSBF, CHOBF, LEOBF HEOBF.		0	0	0
$R.T.\ for$ PSBF		0: 0: 1.576	0: 0: 2.169	0: 0: 3.136
$R.T.\ for$ CHOBF		0: 0: 1.294	0: 0: 2.184	0: 0: 1.981
$R.T.\ for$ LEOBF		0: 0: 1.514	0: 0: 5.101	0: 0: 4.524
$R.T.\ for$ HEOBF		0: 0: 1.170	0: 0: 4.556	0: 0: 4.212

Table (2a) The results for the approximate solution $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ of Example 2, by using **PSBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.01105170918	0.01105159723	0.01105170740	0.01105170918
(0.2,0.2)	0.04885611032	0.04885111124	0.04885604879	0.04885611039
(0.3,0.3)	0.1214872927	0.1214381260	0.1214867479	0.1214872931
(0.4,0.4)	0.2386919517	0.2384355598	0.2386890485	0.2386919534
(0.5,0.5)	0.4121803178	0.4112413324	0.4121684264	0.4121803221
(0.6,0.6)	0.6559627680	0.6532200323	0.6559218660	0.6559627726
(0.7,0.7)	0.9867388264	0.9798945838	0.9866158938	0.9867388017
(0.8,0.8)	1.424346194	1.409137914	1.424015569	1.424346005
(0.9,0.9)	1.992278520	1.961364621	1.991468623	1.992277694
(1,1)	2.718281828	2.659722638	2.716448961	2.718278986
$L.S.E.$		$4.671455778 \times 10^{-3}$	$4.141585486 \times 10^{-6}$	$8.795543580 \times 10^{-12}$
$R.T.$		0: 0: 1.451	0: 0: 2.324	0: 0: 6.225

Table (2b1) The results for the approximate solution $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ of Example 2, by using **CHOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	-2×10^{-10}	-1×10^{-10}
(0.1,0.1)	0.01105170918	0.01105159723	0.0110517071	0.0110517090
(0.2,0.2)	0.04885611032	0.04885111124	0.0488560485	0.0488561102
(0.3,0.3)	0.1214872927	0.1214381260	0.1214867478	0.1214872930
(0.4,0.4)	0.2386919517	0.2384355597	0.2386890480	0.2386919530
(0.5,0.5)	0.4121803178	0.4112413325	0.4121684261	0.4121803217
(0.6,0.6)	0.6559627680	0.6532200323	0.6559218659	0.6559627723
(0.7,0.7)	0.9867388264	0.9798945837	0.9866158936	0.9867388008
(0.8,0.8)	1.424346194	1.409137915	1.424015568	1.424346003
(0.9,0.9)	1.992278520	1.961364621	1.991468623	1.992277693
(1,1)	2.718281828	2.659722638	2.716448958	2.718278982
$L.S.E.$		$4.671455799 \times 10^{-3}$	$4.141595651 \times 10^{-6}$	$8.820482479 \times 10^{-12}$
$R.T.$		0: 0: 1.763	0: 0: 2.559	0: 0: 7.722

Table (2b2) The results for the approximate solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ of Example 2, by using **LEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	1.2×10^{-10}	1×10^{-10}
(0.1,0.1)	0.01105170918	0.01105159723	0.01105170750	0.01105170916
(0.2,0.2)	0.04885611032	0.04885111120	0.04885604884	0.04885611036
(0.3,0.3)	0.1214872927	0.1214381260	0.1214867479	0.1214872931
(0.4,0.4)	0.2386919517	0.2384355597	0.2386890485	0.2386919531
(0.5,0.5)	0.4121803178	0.4112413325	0.4121684264	0.4121803220
(0.6,0.6)	0.6559627680	0.6532200323	0.6559218660	0.6559627724
(0.7,0.7)	0.9867388264	0.9798945837	0.9866158937	0.9867388009
(0.8,0.8)	1.424346194	1.409137914	1.424015569	1.424346003
(0.9,0.9)	1.992278520	1.961364621	1.991468624	1.992277693
(1,1)	2.718281828	2.659722638	2.716448961	2.718278981
$L.S.E.$		4.671455829×10^{-3}	4.141581427×10^{-6}	$8.818977829\times10^{-12}$
$R.T.$		0: 0: 1.966	0: 0: 4.383	0: 0: 10.172

Table (2b3) The results for the approximate solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ of Example 2, by using **HEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	-3×10^{-11}	-2×10^{-10}
(0.1,0.1)	0.01105170918	0.01105159722	0.0110517072	0.01105170878
(0.2,0.2)	0.04885611032	0.04885111125	0.0488560483	0.04885611032
(0.3,0.3)	0.1214872927	0.1214381260	0.1214867478	0.1214872929
(0.4,0.4)	0.2386919517	0.2384355597	0.2386890481	0.2386919531
(0.5,0.5)	0.4121803178	0.4112413325	0.4121684262	0.4121803225
(0.6,0.6)	0.6559627680	0.6532200323	0.6559218662	0.6559627730
(0.7,0.7)	0.9867388264	0.9798945837	0.9866158940	0.9867388023
(0.8,0.8)	1.424346194	1.409137915	1.424015569	1.424346005
(0.9,0.9)	1.992278520	1.961364621	1.991468624	1.992277694
(1,1)	2.718281828	2.659722638	2.716448962	2.718278985
$L.S.E.$		4.671455799×10^{-3}	4.141584907×10^{-6}	$8.828912744\times10^{-12}$
$R.T.$		0: 0: 1.435	0: 0: 2.854	0: 0: 6.458

Table (3a) The results for the approximate solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ of Example 3, by using **PSBF**, **CHOBF**, **LEOBF** and **HEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.03	0.03	0.03	0.03
(0.2,0.2)	0.12	0.12	0.12	0.12
(0.3,0.3)	0.27	0.27	0.27	0.27
(0.4,0.4)	0.48	0.48	0.48	0.48
(0.5,0.5)	0.75	0.75	0.75	0.75
(0.6,0.6)	1.08	1.08	1.08	1.08
(0.7,0.7)	1.47	1.47	1.47	1.47
(0.8,0.8)	1.92	1.92	1.92	1.92
(0.9,0.9)	2.43	2.43	2.43	2.43
(1,1)	3	3	3	3
$\boldsymbol{L.S.E.}$ **for** PSBF, CHOBF, LEOBF, HEOBF		**0**	0	0
$R.T.$ for PSBF		0: 0: 1.123	0: 0: 2.169	0: 0: 4.992
$R.T.$ for CHOBF		0: 0: 1.155	0: 0: 1.450	0: 0: 1.233
$R.T.$ for LEOBF		0: 0: 2.870	0: 0: 2.231	0: 0: 2.512
$R.T.$ for HEOBF		0: 0: 7.144	0: 0: 6.927	0: 0: 6.224

Table (4a) the results for the approximate solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ of Example 4, by using **PSBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.1105170918	0.1104382932	0.1105149002	0.1105170884
(0.2,0.2)	0.2442805516	0.2441075606	0.2442760844	0.2442805448
(0.3,0.3)	0.4049576424	0.4046053655	0.4049504845	0.4049576316
(0.4,0.4)	0.5967298792	0.5959291147	0.5967180279	0.5967298642
(0.5,0.5)	0.8243606355	0.8224759816	0.8243366346	0.8243606134
(0.6,0.6)	1.093271280	1.089042827	1.093213217	1.093271239
(0.7,0.7)	1.409626895	1.400826123	1.409480671	1.409626782
(0.8,0.8)	1.780432742	1.763421872	1.780080879	1.780432384
(0.9,0.9)	2.213642800	2.182825532	2.212853704	2.213641701
(1,1)	2.718281828	2.665431936	2.716631982	2.718278717
$L.S.E.$		$4.131871570 \times 10^{-3}$	$3.494018049 \times 10^{-6}$	$1.103741914 \times 10^{-11}$
$R.T.$		0: 0: 1.139	0: 0: 2.262	0: 0: 4.804

Table (4b1) the results for the approximate solution $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t})$ of Example 4, by using **CHOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.1105170918	0.1104382931	0.1105149001	0.1105170884
(0.2,0.2)	0.2442805516	0.2441075606	0.2442760844	0.2442805447
(0.3,0.3)	0.4049576424	0.4046053655	0.4049504845	0.4049576316
(0.4,0.4)	0.5967298792	0.5959291147	0.5967180275	0.5967298641
(0.5,0.5)	0.8243606355	0.8224759817	0.8243366339	0.8243606134
(0.6,0.6)	1.093271280	1.089042828	1.093213216	1.093271238
(0.7,0.7)	1.409626895	1.400826123	1.409480669	1.409626782
(0.8,0.8)	1.780432742	1.763421872	1.780080878	1.780432384
(0.9,0.9)	2.213642800	2.182825532	2.212853704	2.213641700
(1,1)	2.718281828	2.665431936	2.716631982	2.718278716
$L.S.E.$		$4.131871567 \times 10^{-3}$	$3.494016520 \times 10^{-6}$	$1.103594789 \times 10^{-11}$
$R.T.$		0: 0: 1.654	0: 0: 2.262	0: 0: 5.631

Table (4b2) the results for the approximate solution $u(x,t)$ of Example 4, by using **LEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.1105170918	0.1104382931	0.1105149001	0.1105170884
(0.2,0.2)	0.2442805516	0.2441075607	0.2442760844	0.2442805447
(0.3,0.3)	0.4049576424	0.4046053655	0.4049504846	0.4049576313
(0.4,0.4)	0.5967298792	0.5959291149	0.5967180279	0.5967298642
(0.5,0.5)	0.8243606355	0.8224759818	0.8243366350	0.8243606132
(0.6,0.6)	1.093271280	1.089042827	1.093213216	1.093271237
(0.7,0.7)	1.409626895	1.400826123	1.409480670	1.409626781
(0.8,0.8)	1.780432742	1.763421872	1.780080878	1.780432384
(0.9,0.9)	2.213642800	2.182825533	2.212853704	2.213641699
(1,1)	2.718281828	2.665431936	2.716631982	2.718278716
$L.S.E.$		$4.131871507 \times 10^{-3}$	$3.494018260 \times 10^{-6}$	$1.104057659 \times 10^{-11}$
$R.T.$		0: 0: 1.528	0: 0: 2.559	0: 0: 5.055

Table (4b3) the results for the approximate solution $\boldsymbol{u(x,t)}$ of Example 4, by using **HEOBF**.

(x,t)	Exact Solution	Approximate Solution after using		
		3 terms	5 terms	8 terms
(0,0)	0	0	0	0
(0.1,0.1)	0.1105170918	0.1104382931	0.1105149002	0.1105170884
(0.2,0.2)	0.2442805516	0.2441075606	0.2442760843	0.2442805447
(0.3,0.3)	0.4049576424	0.4046053654	0.4049504844	0.4049576317
(0.4,0.4)	0.5967298792	0.5959291148	0.5967180277	0.5967298638
(0.5,0.5)	0.8243606355	0.8224759818	0.8243366344	0.8243606131
(0.6,0.6)	1.093271280	1.089042827	1.093213216	1.093271238
(0.7,0.7)	1.409626895	1.400826123	1.409480669	1.409626782
(0.8,0.8)	1.780432742	1.763421872	1.780080879	1.780432385
(0.9,0.9)	2.213642800	2.182825531	2.212853703	2.213641701
(1,1)	2.718281828	2.665431936	2.716631982	2.718278717
$L.S.E$		$4.131871631 \times 10^{-3}$	$3.494019843 \times 10^{-6}$	$1.103774723 \times 10^{-11}$
$R.T$		0: 0: 1.498	0: 0: 2.278	0: 0: 6.833

Conclusion

The following tables show a comparison between the exact solution and the approximated solution of the illustrative examples, by using PSBF and OBF with different iterations and number of terms depending of the least square error (L.S.E) and running times (R.T).

Form the Tables (1a)-(4b3) we conclude that:

In general, methods which are used in this paper have been proved their effectiveness in solving MVFIEK2's and SVFIEK2's numerically. Because they produce good results and exact solutions if $\boldsymbol{u(x,t)}$ are polynomials.

(i) For solving MVFIEK2's and SVFIEK2's, where $\boldsymbol{u(x,t)}$ are not polynomials by using PSBF ($\boldsymbol{x^i t^j; i,j = 0,1,2,\ldots}$) as a basis function in series solution method; we obtain similar or better approximate solutions in most examples than that obtained by using CHOBF, LEOBF and HEOBF as bases functions. We also noted that PSBF is faster (depending on running times) than CHOBF, LEOBF and HEOBF because it needs minimum numbers of functions and integration evaluations per iterations.

(ii) The error convergence to zero if the number of used terms increases.

References

[1] **Abdou, M. A. and Al-Bugami, A. M.**, Nonlinear Fredholm-Volterra integral equation and its numerical solutions with quadrature methods, *Journal of Advances in Mathematics*, Vol.4, No. 2 (2013), 415-422.

[2] **Ahmed, S. S.**, Numerical solution for Volterra-Fredholm integral equation of the second kind by using least squares technique, *Iraqi Journal of Science*, Vol.52, No.4 (2011), 504-512.

[3] **Banifatewmi, E.; Razzaghi, M. and Yousefi S.**, Two-dimensional Legendre wavelets method to mixed Volterra-Fredholm integral equations, *Journal of Vibration and Control*, Vol.13, No.11 (2007), 1667-1675.

[4] **Bazrafshan, F.; Mahbobi, A. H.; Neyrameh, A.; Sousaraie, A. and Ebrahim, M.**, solving two-dimensional integral equations, *Journal of King Saud University (Science)*, Vol.23 (2011), 111–114.

[5] **Biswal, P. C.,** *Ordinary differential equations*, 2nd Edition, PHI Learning Private Limited, New Delhi, 2012.

[6] **Borzabadi, A. H. and Heidari, M.**, A successive iterative approach for two dimensional nonlinear Volterra-Fredholm integral equations, *Iranian Journal of Numerical Analysis and Optimization*, Vol4, No.1 (2014), 95-104.

[7] **Hacia, L.**, On approximate solution for integral equations of mixed type, *Journal of Applied Mathematics and Mechanics*, Vol.76 (1996), 415-416.

[8] **Kauthen, J. P.**, Continuous time collocation methods for Volterra-Fredholm integral equations, *Numer. Math.*, Vol. 56 (1989), 409-424.

[9] **Maleknejad, K. and Hadizadeh, M.**, A new computational method for Volterra-Fredholm integral equations, *Comp. Math. Appl.*, Vol.37 (1999), 1-8.

[10] **Maleknejad, K. and Sohrabi, S.**, Legendre polynomial solution of nonlinear Volterra-Fredholm integral equations, *IUST International Journal of Engineering Science,* Vol. 19, No.2-5 (2008), 49-52.

[11] **Mirzaee, F. and Piroozfar, S.,** Numerical solution of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions, *Journal of King Saud University (Science)*, Vol. 22 (2010), 185–193.

[12] **Pachpatte, B. G.**, On mixed Volterra-Fredholm type integral equations, *Indian J. Pure Appl. Math.*, Vol.17, No.4 (1986), 488–496.

[13] **Rahman, M.,** *Integral equations and their applications*, WIT press, Southampton, Boston 2007.

[14] **Saeed, R. K.**, Computational methods for solving system of linear Volterra integral and integro-differential equations, *Ph. D. Thesis, University of Salahaddin/Erbil -collage of Science*, 2006.

[15] **Wazwaz, A. M.**, A reliable treatment for mixed Volterra-Fredholm integral equations, *Applied Mathematics and Computation*, Vol.127, No.2-3 (2002), 405–414.

[16] **Wazwaz,** A-M., *Linear and nonlinear integral equations methods and applications*, Beijing and Springer-Verlag Berlin Heidelberg, 2011.

[17] **Yazdani, S. and Hadizadeh, M.**, Piecewise constant bounds for the solution of nonlinear Volterra-Fredholm integral equations, *Computational and Applied Mathematics*, Vol.31, No. 2 (2012), 1–18.

[18] **Zarebnia, M.**, A numerical solution of nonlinear Volterra-Fredholm integral equations, *Journal of Applied Analysis and Computation,* Vol. 3, No.1 (2013), 95–104.