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Bifurcation by Application Fenichel's Theorem to Singularity Perturbed ODEs

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Abstract

In this Paper, we will state a special structure of the singularly perturbed system with changes in time scales, when the perturbation parameter $\epsilon > 0$. That led to a new structure of fast system. In addition, we will study and studying the invariant manifold of flow of the vector field, and normal hyperpolicity of the fast-slow system in order to provide a proof of the perturbation of normally hyperbolic invariant manifolds due to Fenichel's Theorem.

So, we will introduce some basic ideas of the general case of fast-slow system. Also, we introduce an invariant manifold theory that explain the concept of the normal hyperbolicity invariant manifold of fast-slow system when $\epsilon > 0$. Center Manifold Theorem on fast-slow system will have been stated in order to get an invariant manifold of singularity perturbed ODEs system, also, we will mention the connection between the Fenichel's Theorem and the application of the Center Manifold Theorem on fast-slow system, and then we will study bifurcation theory on singularity perturbed ODE when perturbed parameter $\epsilon > 0$.

Introduction

One of important subjects in mathematical theory of nonlinear singularly perturbed systems is the evolution of Fenichel's Theorem on fast system [10].This theorem states the fast flow transverse to C_0dominates to the slow flow on C_0, then C_0perturbs to a nearby slow manifold C_ϵ. Fenichel's Theorem originated by A. Tikhonov**[9]** at the beginning of the 1950s by studying the periodic orbits for the systems of differential equations containing small parameters.In 1960s and 1970s there was researches that made by P. Kokotovic, H. K. Khalil [2] and R. O'Malley [4] by studying the singularity perturbation methods on the system of differential equation by using the application of the engineering. In this Paper, we state a special structure of singularly perturbed system with changing in time scales, and also explain the normally hyperbolic invariant manifolds on the fast system after changing in time scale, and then apply the Central Manifold Theorem on the fast system that due to Fenichel's Theorem, which study the case that the perturbation parameter $\epsilon > 0$.

A. Basic ideas

In fast-slow system we study the case when $\epsilon > 0$ which we have two basic scales to formulate singularity perturbed ODEs; slow time τ and fast time t. We get two kinds of systems:

$$\epsilon \frac{dx}{dt} = \epsilon \dot{x} = f(x, y, \epsilon) \tag{1}$$

$$\frac{dy}{dt} = \dot{y} = g(x, y, \epsilon) \quad (2)$$

where $. = \frac{dx}{dt}$, $(x, y) \in R^m \times R^n$, and $0 < \epsilon \ll 1$ is a small positive parameter known as the singular perturbation parameter, with functions $f: R^m \times R^n \times R \to R^m$, $g: R^m \times R^n \times R \to R^n$are assumed to be sufficiently smooth. Variables x are called fast variables and variables y are called slow variables.
On fast time scale $t = \epsilon\tau$, we have:

$$\acute{x} = \frac{dx}{d\tau} = f(x, y, \epsilon) \quad (3)$$

$$\acute{y} = \frac{dy}{d\tau} = \epsilon g(x, y, \epsilon) \quad (4)$$

Where ' = $\frac{dx}{d\tau}$.
In this paper, we will study the system above form the singularity perturbed ODE system which called the fast system.

Definition 1. [7] *The set*

$$C_0 = \{(x, y) \in R^m \times R^n : f(x, y, 0) = 0\} \quad (5)$$

called a critical set. If C_0 is a submanifold of $R^m \times R^n$, then C_0 is called a critical manifold.

B.Invariant Manifolds [3]

The invariant manifold theory is one of the bases of the geometric approach to multiple time scale dynamics, which will be used in Fenichel's Theorem through studying the flow of the fast system, that depended on perturbation parameter $\epsilon > 0$. In this subsection, the review of invariant manifold will be given and we introduce the definition of invariant manifold as follows:

Definition 2. [1] *Let M be a compact connected C^rmanifold with boundary embedded in R^n. Then M is called an invariant manifold if for every $p \in M$, we have a flow $\emptyset_t(.)$ which defined by vector field such that $\emptyset_t(p) \in M$ for all $t \in R$, M is called a locally invariant manifold if for each $p \in M$, there exists a time interval $I_p(t_1, t_2)$ such that$\emptyset_t(p) \in M$ for all $t \in I_p$.*

C.Normal Hyperbolicty [7]

The notion of a normally hyperbolic manifold is one of the most important concepts in the geometric theory of dynamical systems, which says that a manifold is normally hyperbolic if the linearized flow in the normal direction dominates the linearized flow in the tangential direction. Normal hyperpolicity is an important case in stating the hyperbolic equilibrium point according to the invariant manifold of fast system that due to study Fenichel's Theorem on singularity perturbed system. Here we state the definition of normal hyperpolicity of fast system:

Definition 3. [11] *A subset $M$of $C_0$is called normally hyperbolic,if the Jacobian matrix $(D_x f)(p, 0)$ of first partial derivatives with respect to the fast variables has no eigenvalues with zero real part for all $p \in M$ and $(D_x f)(p, 0)$ is m×m matrix.*

Definition 4. [7] *A normally hyperbolic subset M of C_0 is called attracting if all eigenvalues of $(D_x f)(p,0)$ have negative real part forp $\in M$. Similarly, M is called repelling if all eigenvalues have positive real part. If M is normally hyperbolic and neither attracting nor repelling, it is of saddle type.*

We now introduce Center Manifold Theory of fast system in order to reduce system defined as a flow of singularity perturbed system under the normal hyperbolic invariant manifold.

D. Center Manifold Theorem in Fast System

The Center Manifold Theory study the existence of stable, unstable manifold, unique and possibly non-unique center manifold in the dynamical system**[8]**. Central Manifold Theorem in fast system explain that we can find an invariant manifold under the transformation of fast system to a diagonal form to obtain a flow under invariant manifold, then joining Central Manifold Theorem to Fenichel's Theorem that satisfy normal hyperbolicity of an invariant manifold with the locally invariant manifold. We formulate the main theorems without proof that allow us to reduce the dimension of a given system near a local bifurcation.

Theorem 1.[8] *Consider a fast system:*

$$\frac{dx}{d\tau} = f(x, y, \epsilon), \tag{6}$$

$$\frac{dy}{d\tau} = \epsilon g(x, y, \epsilon), \tag{7}$$

By transform (6), (7) to the diagonal we have:

$$x' = Ax + F(x, y, \epsilon), \tag{8}$$

$$y' = By + G(x, y, \epsilon), \tag{9}$$

$$\epsilon' = 0. \tag{10}$$

where $x \in R^m, y \in R^n,\ \epsilon \in R$ and A, B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. The functions F and G are C^2 such that $F(x, y, \epsilon) = f(x, y, \epsilon), G(x, y, \epsilon) = \epsilon g(x, y, \epsilon)$ with

$$F(0) = G(0) = 0,$$

$$\frac{\partial F}{\partial x}(0) = 0, \qquad \frac{\partial F}{\partial y}(0) = 0,$$

$$\frac{\partial G}{\partial x}(0) = 0, \qquad \frac{\partial G}{\partial y}(0) = 0.$$

Then, there exists an invariant manifold $y = h^\epsilon(x, \epsilon), |x| < m, |\epsilon| < \gamma, \gamma > 0$ with $|h^\epsilon(x)| < C|\epsilon|$, where C is a constant which depends on $A, B, F,$ and G. The value of $h^\epsilon(x)$ is evaluated as follows:

$$Dh^\epsilon(x)[Ax + F(x, h^\epsilon(x), \epsilon)] - Bh^\epsilon(x) - G(x, h^\epsilon(x), \epsilon) = 0. \tag{11}$$

The flow on the invariant manifold is given by the equation

$$\acute{x} = Ax + F(x, h^\epsilon(x), \epsilon), \tag{12}$$

Note that the center manifold of (8), (9), (10) is $y = h^\epsilon(x, \epsilon)$ satisfy the condition:

$$\frac{\partial h^\epsilon(x, \epsilon)}{\partial x}(0,0,0) = 0.$$

The flow on the center manifold is given by:

$$x' = Ax + F(x, h^\epsilon(x), \epsilon), \tag{13}$$

$$\epsilon' = 0. \tag{14}$$

However, the equation which describes the flow along the center manifold will be an ODE with a parameter ϵ.

E.Fenichel's Theorem

Fenichel published his general invariant manifold theory during the 1970s and then applied theory on fast-slow systems in 1979 [6] entitled "*Geometric Singular Perturbation Theory for Ordinary Differential Equations*".

Fenichel's Theorem asserts the existence of a manifold that is a perturbation of the critical sets M_0, it will be connected with the flow of fast system when $\epsilon > 0$ and with the normal hyperbolic invariant manifold according to Central Manifold Theorem, which considers the manifolds, their local stable, and unstable manifolds, to get a reduced system and then applies bifurcation theory on singularity perturbed ODEs when perturbation parameter $\epsilon > 0$.

Theorem 2. [7] ***(Fenichel's Theorem)***

Consider the fast system

$$\frac{dx}{d\tau} = f(x, y, \epsilon), \tag{15}$$

$$\frac{dy}{d\tau} = \epsilon g(x, y, \epsilon). \tag{16}$$

Suppose that $M = M_0$ is a compact normally hyperbolic submanifold of the critical manifold C_0 of (15), (16) and that $f, g \in C^r$ when ($r < \infty$). Then for $\epsilon > 0$ sufficiently small, the following hold:

(1) There exists a locally invariant manifold $M_\epsilon$diffeomorphic to M_0(i.e.) ($M_\epsilon$is smooth and invertible with smooth inverse) and M_ϵ lies within $(\epsilon)$of M_0. Moreover M_ϵ is locally invarient under the flow of (15), (16).
(2) M_ϵ has Hausdorff distance(ϵ) (as $\epsilon \to 0$) from M_0.
(3) The flow on M_ϵ converges to the slow flow as$\epsilon \to 0$.
(4)M_ϵ is C^r-smooth.
(5) M_ϵ is normally hyperbolic and has the same stability properties with respect to the fast variables as M_0 (attracting, repelling, or of saddle type).
(6) M_ϵ is usually not unique. In regions that remain at a fixed distance from M_ϵ.

From Fenichel's Theorem, we get the following result that depends on the invariant manifold of the flow of fast-slow system by using Central Manifold Theorem

Theorem 3: *Consider the fast system (15), (16) that satisfy the conditions*
$(1) f, g \in C^r (r < 1)$,
(2) The set $M_0$is a compact normally hyperbolic sub manifold, and is normally relative to the system

$$x' = f(x, y, 0), \tag{17}$$
$$y' = 0, \tag{18}$$

(3) The set M_0 is given as the graph of the C^r function $h^0(x)$ for $x \in Q$, the set Q is compact.
*If $\epsilon > 0$ is su*fficiently small, there is a function $y = h^\epsilon(x, \epsilon)$, defined on Q, so that the graph

$$M_0 = \{(x, y): y = h^\epsilon(x, \epsilon)\}, \tag{19}$$

is locally invariant under (17), (18). Here $h^\epsilon(x, \epsilon)$ is a C^r, for any $r < 1$, jointly in x and ϵ.
Proof.
We substitute the function $h^\epsilon(x, \epsilon)$into (15), (16) and see that the x equation will decouple from that of $\acute{x}$ equation. Hence, we obtain an equation for the variation of the variable x. Since x parameterizes the manifold M_ϵ, this equation will suffice to describe the flow on M_ϵ, it is given by

$$y' = \epsilon g(x, h^\epsilon(x, \epsilon), \epsilon), \tag{20}$$

By changing the slow scale in (19) we get that

$$\dot{y} = g(x, h^\epsilon(x, \epsilon), \epsilon), \tag{21}$$

Which is represent to system of differential equation.

■

Remark 1. *A manifold M_ϵ, as obtained in the conclusion of Fenichel's Theorem is called a slow manifold.*

In any case, it is important to distinguish between a critical manifold obtained in the singular limit $\epsilon = 0$ and a slow manifold that is obtained via Fenichel's theorem for $\epsilon > 0$, after all, a major theorem has been applied to go from a critical to a slow manifold.

Remark 2. *The system (20) has an advantage as ϵ approaches to 0, which has the following form*

$$\dot{y} = g(h^0(y), y, 0). \tag{22}$$

That it naturally describes a flow on the critical manifold M_0.

Here we give an example to illustrate the applying of Fenichel's Theorem and Central Manifold Theorem on the fast system:
Example 1.[7]. *Consider the (unforced) van der Pol equation:*

$$\epsilon \dot{x} = y - \frac{x^3}{3} + x, \tag{23}$$
$$\dot{y} = -x, \tag{24}$$

with critical manifold

$$C_0 = \left\{(x,y) \in R^2 : y = \frac{x^3}{3} - x\right\}$$

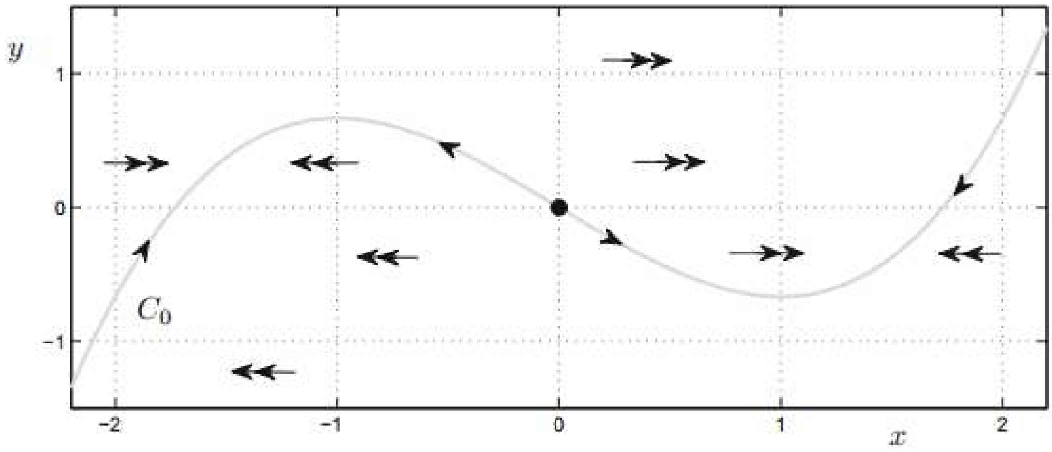

Figure-1: The critical manifold of the van der pol equation.

The singular limit $\epsilon = 0$ *in (23), (24) leads to*

$$0 = y - \frac{x^3}{3} + x,$$
$$\dot{y} = -x.$$

Differentiating the equation $f(x,y) = y - \frac{x^3}{3} + x$, *we obtain*

$$\frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = (1\text{-}x^2)\,\dot{x} + \dot{y} = 0.$$

It follows from (23), (24) that $\dot{y} = -x$, *and*

$$\dot{x} = \frac{x}{1 - x^2}.$$

Let $t = \epsilon\tau$, *we can rewrite (23), (24) in the following equivalent form:*

$$x' = f(x,z), \quad (25)$$
$$z' = \epsilon(f(x,z) - x), \quad (26)$$
$$\epsilon' = 0, \quad (27)$$

where $f(x,z) = z - \frac{x^3}{3}$ *and* $z = y + x$.

By Theorem 1, (25), (26), (27) has a center manifold $z = h^{\epsilon}(x)$. *To find an approximation to the function h set*

$$\emptyset(x,\epsilon) = \frac{\partial \emptyset}{\partial x}(x,\epsilon) f(x,\emptyset) + \emptyset(x,\epsilon) - \epsilon\big(f\big(x,\emptyset(x,\epsilon)\big) - x\big)$$

if $\emptyset(x,\epsilon) = \frac{1}{1-x^2}$ *then*

$$h^{\epsilon}(x) = -x + \frac{x^3}{3} + (|x^4| + |\epsilon^4|).$$

We have derived an explicit form for the slow flow on the critical manifold C_0. *Note carefully that the above ODE has an unstable equilibrium at* $x = 0$ *and is not defined at the points* $x = 1$ *and* $x = -1$. *In fact, the points* $x = \pm 1$ *split the critical manifold into three parts*

$$C_0^{a-} = \{C_0 \cap \{(x,y) \in R^2 : x < -1\}\},$$
$$C_0^{r} = \{C_0 \cap \{(x,y) \in R^2 : -1 < x < 1\}\},$$
$$C_0^{a+} = \{C_0 \cap \{(x,y) \in R^2 : x > 1\}\}.$$

Where $C_0^{a\pm}$ *are normally hyperbolic attracting and* C_0^{r} *is normally hyperbolic repelling.*

F. Bifurcation Theory in Singularity Perturbed ODEs

Consider the system of ordinary differential equations

$$z' = F(x, z), \tag{28}$$
$$F(0, \epsilon) \equiv 0, \tag{29}$$

where $z \in R^{n+m}, \epsilon$ is a p-dimensional parameter. We say that $\epsilon = 0$ is a bifurcation point for (28) if there exists two points ϵ_1 and ϵ_2 such that the local phase portraits of (28) for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$are not topologically equivalent.

Assume the linearization of (28) as about $z = 0$is:

$$z' = L(\epsilon)z. \tag{30}$$

If all eigenvalues of $L(0)$have non-zero real parts then, $\epsilon = 0$ is not a bifurcation point. Since a local bifurcation theory happened only when the linearized system has eigenvalues with zero real parts [5]**.** We assume that $L(0)$ has m eigenvalues with zero real parts and n eigenvalues with negative real part, with supposing that $L(0)$does not have any positive eigenvalues then, the system (28) can rewrite as:

$$x' = Ax + F(x, y, \epsilon), \tag{31}$$
$$y' = By + G(x, y, \epsilon), \tag{32}$$
$$\epsilon' = 0, \tag{33}$$

where $x \in R^m, y \in R^n$and A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. From Central Manifold Theorem, there exist an invariant manifold $y = h^\epsilon(x)$, and the flow on the invariant manifold is given as:

$$x' = Ax + F(x, h^\epsilon(x), \epsilon), \tag{34}$$
$$\epsilon' = 0. \tag{35}$$

We can apply bifurcation problems on fast system according to Centre Manifold Theory with Fenichel's theorem.

G. Fold Bifurcation in Singularity Perturbed ODEs

A fold bifurcation point is a pair of equilibria that meets and disappears with a zero eigenvalue [8]. One of the equilibria (saddle) is unstable while the other (node) is stable.

In this section, we study fold bifurcation of the singularity perturbed system by applying Center Manifold Theorem according to the conditions of Feinchel'sTheorem, with a bifurcation parameter $\epsilon = 0$ and the non-hyperbolic equilibrium points at $x = 0, y = 0, \epsilon = 0$.

Consider the fast system:

$$\frac{dx}{d\tau} = f(x, y, \epsilon), \tag{36}$$
$$\frac{dy}{d\tau} = \epsilon g(x, y, \epsilon), \tag{37}$$

where $x \in R^m$and $y \in R^n$, x is a slow variable and y is a fast variable, $0 < \epsilon \ll 1$. When $\epsilon > 0$ we have a reduced system by applying Center Manifold Theorem with Fenichel's Theorem which gives

$$x' = Ax + F(x, h^\epsilon(x, \epsilon), \epsilon), \tag{38}$$
$$\epsilon' = 0, \tag{39}$$

where $F: R^m \times R^n \times R \to R^m$,ϵ is the bifurcation parameter, and the reduced system satisfy the non-hyperbolic condition

$$\frac{\partial F}{\partial x}(0,0,0) = 0,$$

and

$$\frac{\partial F}{\partial y}(0,0,0) = 0.$$

Now suppose K be a set of all equilibrium points $(x^*, (h^\epsilon(x))^*, \epsilon^*)$ that defines to be

$$K = \{(x, h^\epsilon(x, \epsilon), \epsilon) \in R^m \times R^n \times R: \mathrm{F}(x^*, (h^\epsilon(x))^*, \epsilon^*) = 0\},$$

and we define the function H by

$$H(x^*, (h^\epsilon(x))^*, \epsilon^*) = F(x^*, (h^\epsilon(x))^*, \epsilon^*) \tag{40}$$

Theorem 4. *Consider fast system*

$$\frac{dx}{d\tau} = x' = f(x, y, \epsilon), \tag{41}$$

$$\frac{dy}{d\tau} = y' = \epsilon g(x, y, \epsilon) \tag{42}$$

defined on set of critical points K with the non-hyperbolic conditions. If the following conditions are holds:

(1) $\frac{\partial F}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) \neq 0$,

(2) $\frac{\partial^2 F}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*) \neq 0$,

then for $\epsilon > 0$ *the system (38), (39) undergo to fold bifurcation point when* $(x^*, (h^\epsilon(x))^*, \epsilon^*)$ *closes to* $(0,0,0)$ *and the flow of the singularity perturbed ODEs (41), (42) on a center manifold is locally equivalent to one of the following normal forms*

$$\frac{d\eta}{dt} = \pm\mu \pm \eta^2$$

Where μ *is a bifurcation parameter and it is sign is the same sign of* $\frac{\partial f}{\partial \epsilon}$, $\eta^2 = |a(\mu)|\xi, \xi$ *is a variable contain* x, *the sign of* η^2 *is the same sign of* $\frac{\partial^2 f}{\partial x^2}$.

Proof.

For the first part, we differentiate $H(x^*, (h^\epsilon(x))^*, \epsilon^*)$ with respect to ϵ we obtain

$$\frac{\partial H}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) = \frac{\partial F}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial x}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial F}{\partial h^\epsilon(x,\epsilon)}(x^*, (h^\epsilon(x))^*, \epsilon^*)$$
$$\frac{\partial h^\epsilon(x,\epsilon)}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial F}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau}.$$

Evaluate above equation at (0,0,0) and apply the non-hyperbolic condition $\frac{\partial F}{\partial x}(0) = 0$ *and* $\frac{\partial F}{\partial h^\epsilon(x,\epsilon)}(0) = 0$ with condition (1), it is easy to see that

$$\frac{\partial H}{\partial \epsilon}(0,0,0) = \frac{\partial F}{\partial x}(0,0,0)\frac{\partial x}{\partial \epsilon}(0,0,0) + \frac{\partial F}{\partial h^\epsilon(x,\epsilon)}(0,0,0)\frac{\partial h^\epsilon(x,\epsilon)}{\partial \epsilon}(0,0,0) + \frac{\partial F}{\partial \epsilon}(0,0,0) = \frac{\partial F}{\partial \epsilon}(0,0,0) \neq 0.$$

Then,

$$\frac{\partial H}{\partial \epsilon}(0,0,0) \neq 0.$$

Now we prove the second part

Differentiate $H(x^*, (h^\epsilon(x))^*, \epsilon^*)$ with respect to x we have:

$$\frac{\partial H}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) = \frac{\partial F}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial F}{\partial h^\epsilon(x,\epsilon)}(x^*, (h^\epsilon(x))^*, \epsilon^*)$$
$$\frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial F}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*).$$

The second differentiation of the above equation with respect to x is as follows:

$$\frac{\partial^2 H}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*) = \frac{\partial}{\partial x}(\frac{\partial F}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial F}{\partial h^\epsilon(x,\epsilon)}(x^*, (h^\epsilon(x))^*, \epsilon^*)$$
$$\frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial F}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*))$$
$$= \frac{\partial^2 F}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial F}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{d^2 x}{d\tau^2} + \frac{\partial F}{\partial h^\epsilon(x,\epsilon)}$$
$$(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial^2 h^\epsilon(x,\epsilon)}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial^2 F}{\partial h^\epsilon(x,\epsilon)\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)$$

$$\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+\frac{\partial F}{\partial \epsilon}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial^2 \epsilon}{\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+$$
$$\frac{\partial^2 F}{\partial \epsilon \partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*),$$

evaluate at $(0,0,0)$and substitute condition(1), and (2) we get:

$$\frac{\partial^2 H}{\partial x^2}(0,0,0) == \frac{\partial^2 F}{\partial x^2}(0,0,0)\frac{dx}{d\tau}+\frac{\partial^2 F}{\partial h^{\epsilon}(x,\epsilon)\partial x}(0,0,0)\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(0,0,0)+\frac{\partial F}{\partial \epsilon}(0,0,0)\frac{\partial^2 \epsilon}{\partial x^2}(0,0,0)+$$
$$\frac{\partial^2 F}{\partial \epsilon \partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\neq 0,$$

then

$$\frac{\partial^2 H}{\partial x^2}(0,0,0)\neq 0.$$

Then $(0,0,0)$is a fold bifurcation point and it is locally equivalent to one of the following normal forms

$$\frac{d\eta}{dt}=\pm\mu\pm\eta^2.$$

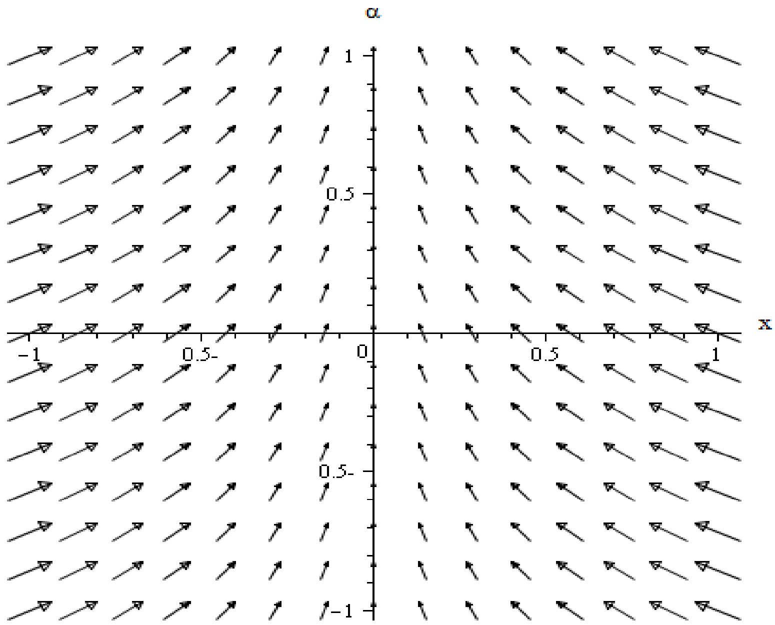

Figure-2: The vector field direction for the normal form of fold bifurcation.

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H. Pitchfork bifurcation in Singularity Perturbed ODEs

In the pitchfork bifurcation, an equilibrium point reverses its stability, and two new equilibrium points are born [5].

In this Section, we explain the concept of pitchfork bifurcation of sing fast system by applying singularity perturbed ODEs when$\epsilon > 0$.

Consider a fast system given by (41), (42). Define

$$f(x,y,\epsilon)=xf_1(x,y,\epsilon),$$
$$\epsilon g(x,y,\epsilon)=xf_2(x,y,\epsilon),$$

where $f_1, f_2$are functions, $f_1: R^m\times R^n\times R\to R^m$and $f_2: R^m\times R^n\times R\to R^n$,$x\in R^m$ and$y\in R^n$, $\epsilon\in R$.
When $\epsilon > 0$ we have a reduced system by applying Central Manifold Theorem with Fenichel's Theorem as follows:

$$x' = Ax + xU(x,h^{\epsilon}(x,\epsilon),\epsilon), \quad (43)$$
$$\epsilon' = 0. \quad (44)$$

Define

$$H(x, h^{\epsilon}(x,\epsilon), \epsilon) = Ax + F(x, h^{\epsilon}(x,\epsilon), \epsilon),$$

with a non-hyperbolic equilibrium at $x^* = 0, (h^{\epsilon}(x))^* = 0, \epsilon^* = 0$that satisfy the condition

$$\frac{\partial U}{\partial \epsilon}(0,0,0) = 0.$$

Suppose K_1 be a set of all equilibrium points $(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$defines as follows:

$$K_1 = \{(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \in R^m \times R^n \times R: U(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0\}.$$

The pitchfork bifurcation theorem on singularity perturbed ODEs introduce as follows:

Theorem 5.*Consider the fast system (43), (44) defined on the set of critical points K_1. If the following conditions are holds:*

(1) $\frac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0,$

(2) $\frac{\partial^2 U}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \neq 0, \frac{\partial^2 U}{\partial x^2}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0,$

(3) $\frac{\partial^3 U}{\partial x^3}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \neq 0,$

then for $\epsilon > 0$ the system (43), (44)undergo to pitchfork bifurcation point when $(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$closes to $(0,0,0)$ and the flow of the singularity perturbed ODEs (41), (42) on a center manifold is locally equivalent to one of the following normal forms

$$\frac{d\eta}{dt} = \pm\mu\eta \pm \eta^3.$$

Where μ is a bifurcation parameter and it is sign is the same sign of $\frac{\partial^2 U}{\partial x \partial \epsilon}$, $\eta^2 = |a(\mu)|\xi, \xi$ is a variable contain x, the sign of η^2 is the same sign of $\frac{\partial^3 U}{\partial x^3}$.

Proof.

For the first condition

$$\frac{\partial H}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = x(\frac{\partial U}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$$
$$\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) + \frac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)) +$$
$$U(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau},$$

differentiate the above equation with respect to ϵ as follows:

$$\frac{\partial^2 H}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = \frac{\partial}{\partial \epsilon}(x(\frac{\partial U}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$$
$$\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) + \frac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)) +$$
$$U(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau}),$$
$$= x(\frac{\partial^2 U}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial^2 U}{\partial h^{\epsilon}(x,\epsilon)\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$$
$$\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) + \frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$$
$$+ \frac{\partial^2 U}{\partial \epsilon^2}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$$
$$+ \frac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{\partial^2 \epsilon}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)) + \frac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*)\frac{dx}{d\tau},$$

evaluate at $(0,0,0)$and apply condition (1), and (2) we have:

$$\frac{\partial^2 H}{\partial x\partial\epsilon}(0,0,0)=x\frac{\partial^2 U}{\partial x\partial\epsilon}(0,0,0)\frac{dx}{d\tau}+\frac{\partial^2 U}{\partial h^{\epsilon}(x,\epsilon)\partial\epsilon}(0,0,0)\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(0,0,0)$$
$$+\frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(0,0,0)\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x\partial\epsilon}(0,0,0)+\frac{\partial^2 U}{\partial\epsilon^2}\big((0,0,0)\big)\frac{\partial\epsilon}{\partial x}(0,0,0)\neq 0,$$

then

$$\frac{\partial^2 H}{\partial x\partial\epsilon}(0,0,0)\neq 0.$$

For the second property

$$\frac{\partial^3 H}{\partial x^3}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)=x(\frac{\partial^3 U}{\partial x^3}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{dx}{d\tau}+2\frac{\partial^2 U}{\partial x^3}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{d^2x}{d\tau^2}+\frac{\partial U}{\partial x}$$
$$(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{d^3x}{d\tau^3}+\frac{\partial^3 U}{\partial h^{\epsilon}(x,\epsilon)\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+$$
$$2\frac{\partial^2 U}{\partial h^{\epsilon}(x,\epsilon)\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+\frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)$$
$$\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x\partial\epsilon}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial^3 h^{\epsilon}(x,\epsilon)}{\partial x^3}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+\frac{\partial^3 U}{\partial\epsilon\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial\epsilon}{\partial x}$$
$$(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+2\frac{\partial^2 U}{\partial\epsilon\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{\partial^2\epsilon}{\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)+\frac{\partial U}{\partial\epsilon}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)$$
$$\frac{\partial^3\epsilon}{\partial x^3}(x^*,(h^{\epsilon}(x))^*,\epsilon^*))+\frac{\partial^2 U}{\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{dx}{d\tau}+\frac{\partial U}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{d^2x}{d\tau^2}+\frac{\partial U}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)$$
$$\frac{d^3x}{d\tau^3}+\frac{\partial^2 U}{\partial x^2}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{d^2x}{d\tau^2}+\frac{\partial U}{\partial x}(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{dx}{d\tau}+U(x^*,(h^{\epsilon}(x))^*,\epsilon^*)$$
$$\frac{d^3x}{d\tau^3})+U(x^*,(h^{\epsilon}(x))^*,\epsilon^*)\frac{dx}{d\tau},$$

evaluate the above equation at $(0,0,0)$and apply conditions (1), (2), and (3) we get:

$$\frac{\partial^3 H}{\partial x^3}(0,0,0)=x(\frac{\partial^3 U}{\partial x^3}(0,0,0)\frac{dx}{d\tau}+2\frac{\partial^2 U}{\partial x^3}(0,0,0)\frac{d^2x}{d\tau^2}+\frac{\partial U}{\partial x}(0,0,0)\frac{d^3x}{d\tau^3}+$$
$$\frac{\partial^3 U}{\partial h^{\epsilon}(x,\epsilon)\partial x^2}(0,0,0)\frac{\partial h^{\epsilon}(x,\epsilon)}{\partial x}(0,0,0)+2\frac{\partial^2 U}{\partial h^{\epsilon}(x,\epsilon)\partial x}(0,0,0)\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x^2}(0,0,0)+$$
$$\frac{\partial U}{\partial h^{\epsilon}(x,\epsilon)}(0,0,0)\frac{\partial^2 h^{\epsilon}(x,\epsilon)}{\partial x\partial\epsilon}(0,0,0)\frac{\partial^3 h^{\epsilon}(x,\epsilon)}{\partial x^3}(0,0,0)+\frac{\partial^3 U}{\partial\epsilon\partial x^2}(0,0,0)\frac{\partial\epsilon}{\partial x}(0,0,0)$$
$$+2\frac{\partial^2 U}{\partial\epsilon\partial x}(0,0,0)\frac{\partial^2\epsilon}{\partial x^2}(0,0,0))+\frac{\partial^2 U}{\partial x^2}(0,0,0)\frac{dx}{d\tau}+\frac{\partial U}{\partial x}(0,0,0)\frac{d^2x}{d\tau^2}+\frac{\partial U}{\partial x}(0,0,0)$$
$$\frac{d^3x}{d\tau^3}+\frac{\partial U}{\partial x}(0,0,0)\frac{dx}{d\tau}+U(0,0,0)\frac{d^3x}{d\tau^3})+U(0,0,0)\frac{dx}{d\tau}\neq 0$$

then

$$\frac{\partial^3 H}{\partial x^3}(0,0,0)\neq 0$$

then, the equilibrium point close to $(0,0,0)$and it is locally equivalent to one of the following normal forms

$$\frac{d\eta}{dt}=\pm\mu\eta\pm\eta^3.$$

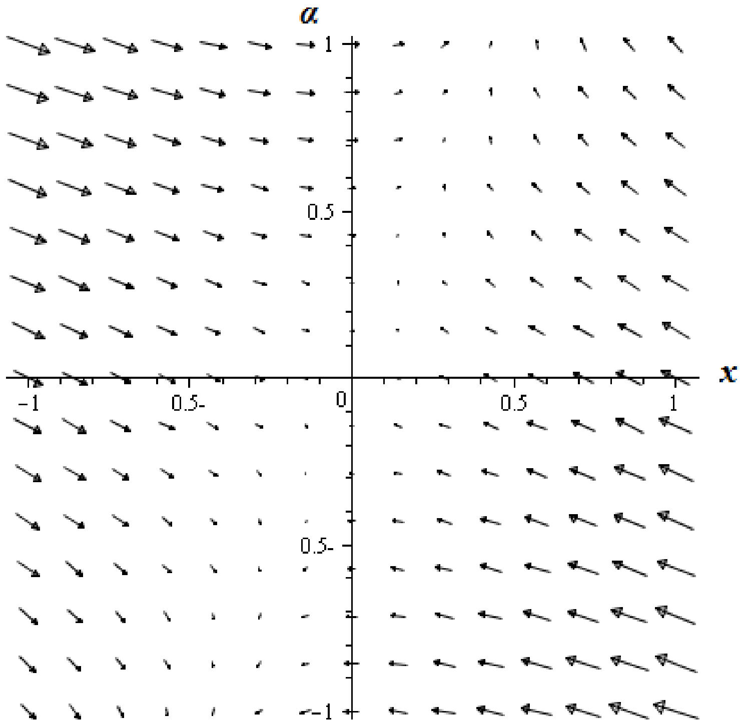

Figure-3: The vector field direction for the normal form of pitchfork bifurcation.

■

I. Transcritical Bifurcation in Singularity Perturbed ODEs

A transcritical bifurcation is one in which an equilibrium point exists for all values of a parameter and is never destroyed [8]. In transcritical bifurcation there is an exchange of stability between two equilibrium points; one is unstable and the another is stable equilibrium point.

In this section, we apply the transcritical bifurcation on singularity perturbed ODEs when$\epsilon > 0$.

Consider a fast system given by (41), (42). Define:

$$f(x, y, \epsilon) = x f_1(x, y, \epsilon),$$
$$\epsilon g(x, y, \epsilon) = x f_2(x, y, \epsilon)$$

where f_1, f_2 are functions, $f_1: R^m \times R^n \times R \to R^m$and $f_2: R^m \times R^n \times R \to R^n$, $x \in R^m$ and $y \in R^n$, $\epsilon \in R$.

When $\epsilon > 0$ we have a reduced system by applying Central Manifold Theorem with Fenichel's Theorem as follows:

$$x' = Ax + xU(x, h^{\epsilon}(x, \epsilon), \epsilon), \tag{45}$$
$$\epsilon' = 0. \tag{46}$$

Define

$$H(x, h^{\epsilon}(x, \epsilon), \epsilon) = Ax + F(x, h^{\epsilon}(x, \epsilon), \epsilon),$$

with a non-hyperbolic equilibrium at $x^* = 0, (h^{\epsilon}(x))^* = 0, \epsilon^* = 0$ that satisfy the condition

$$U(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0.$$

Suppose K_2 be a set of all equilibrium points $(x^*, (h^{\epsilon}(x))^*, \epsilon^*)$defines as follows:

$$K_2 = \{(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \in R^m \times R^n \times R: U(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0\}.$$

The transcritical bifurcation theorem on singularity perturbed ODEs is as follows:

Theorem 6.*Consider the fast system (45), (46) defined on the set of critical points K_2. If the following conditions are holds:*

(1) $\dfrac{\partial U}{\partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) = 0,$

(2) $\dfrac{\partial^2 U}{\partial x \partial \epsilon}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \neq 0, \dfrac{\partial^2 U}{\partial x^2}(x^*, (h^{\epsilon}(x))^*, \epsilon^*) \neq 0,$

then for $\epsilon > 0$ *the system (45), (46)undergo to transcritical bifurcation point when* $(x^*, (h^\epsilon(x))^*, \epsilon^*)$ *closes to* $(0,0,0)$ *and the flow of the singularity perturbed ODEs (41), (42) on a center manifold is locally equivalent to one of the following normal forms*

$$\frac{d\eta}{dt} = \pm\mu\eta \pm \eta^2.$$

Where μ *is a bifurcation parameter and it is sign is the same sign of* $\frac{\partial U}{\partial \epsilon}$, $\eta^2 = |a(\mu)|\xi, \xi$ *is a variable contain* x, *the sign of* η^2 *is the same sign of* $\frac{\partial U}{\partial x}$.

Proof.

For the first property

$$\frac{\partial^2 H}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*) = x(\frac{\partial^2 U}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial U}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{d^2x}{d\tau^2} + \frac{\partial^2 U}{\partial h^\epsilon(x,\epsilon)\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial U}{\partial h^\epsilon(x,\epsilon)}(x^*, (h^\epsilon(x))^*, \epsilon^*) \frac{\partial^2 h^\epsilon(x,\epsilon)}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial^2 U}{\partial \epsilon\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial U}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial^2 \epsilon}{\partial x^2}(x^*, (h^\epsilon(x))^*, \epsilon^*)) + \frac{\partial U}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + U(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{d^2x}{d\tau^2},$$

evaluate at $(0,0,0)$and apply condition (1) we have:

$$\frac{\partial^2 H}{\partial x^2}(0,0,0) = x(\frac{\partial^2 U}{\partial x^2}(0,0,0)\frac{dx}{d\tau} + \frac{\partial U}{\partial x}(0,0,0)\frac{d^2x}{d\tau^2} + \frac{\partial^2 U}{\partial h^\epsilon(x,\epsilon)\partial x}(0,0,0) \frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(0,0,0) + \frac{\partial U}{\partial h^\epsilon(x,\epsilon)}(0,0,0)\frac{\partial^2 h^\epsilon(x,\epsilon)}{\partial x^2}(0,0,0) + \frac{\partial^2 U}{\partial \epsilon\partial x}(0,0,0)\frac{\partial \epsilon}{\partial x}(0,0,0)) \neq 0,$$

then

$$\frac{\partial^2 H}{\partial x^2}(0,0,0) \neq 0.$$

For the second property

$$\frac{\partial^2 H}{\partial x\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) = x(\frac{\partial^2 U}{\partial x\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau} + \frac{\partial^2 U}{\partial h^\epsilon(x,\epsilon)\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) \frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial U}{\partial h^\epsilon(x,\epsilon)}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial^2 h^\epsilon(x,\epsilon)}{\partial x\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial^2 U}{\partial \epsilon^2}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial \epsilon}{\partial x}(x^*, (h^\epsilon(x))^*, \epsilon^*) + \frac{\partial U}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{\partial^2 \epsilon}{\partial x\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)) + \frac{\partial U}{\partial \epsilon}(x^*, (h^\epsilon(x))^*, \epsilon^*)\frac{dx}{d\tau},$$

evaluate at $(0,0,0)$and apply condition (2) we have:

$$\frac{\partial^2 H}{\partial x\partial \epsilon}(0,0,0) = x(\frac{\partial^2 U}{\partial x\partial \epsilon}(0,0,0)\frac{dx}{d\tau} + \frac{\partial^2 U}{\partial h^\epsilon(x,\epsilon)\partial \epsilon}(0,0,0)\frac{\partial h^\epsilon(x,\epsilon)}{\partial x}(0,0,0) + \frac{\partial U}{\partial h^\epsilon(x,\epsilon)}(0,0,0)\frac{\partial^2 h^\epsilon(x,\epsilon)}{\partial x\partial \epsilon}(0,0,0) + \frac{\partial^2 U}{\partial \epsilon^2}(0,0,0)\frac{\partial \epsilon}{\partial x}(0,0,0)) \neq 0,$$

then

$$\frac{\partial^2 H}{\partial x\partial \epsilon}(0,0,0) \neq 0.$$

then, the equilibrium point close to (0,0,0)and it is locally equivalent to one of the following normal forms:

$$\frac{d\eta}{dt} = \pm\mu\eta \pm \eta^2$$

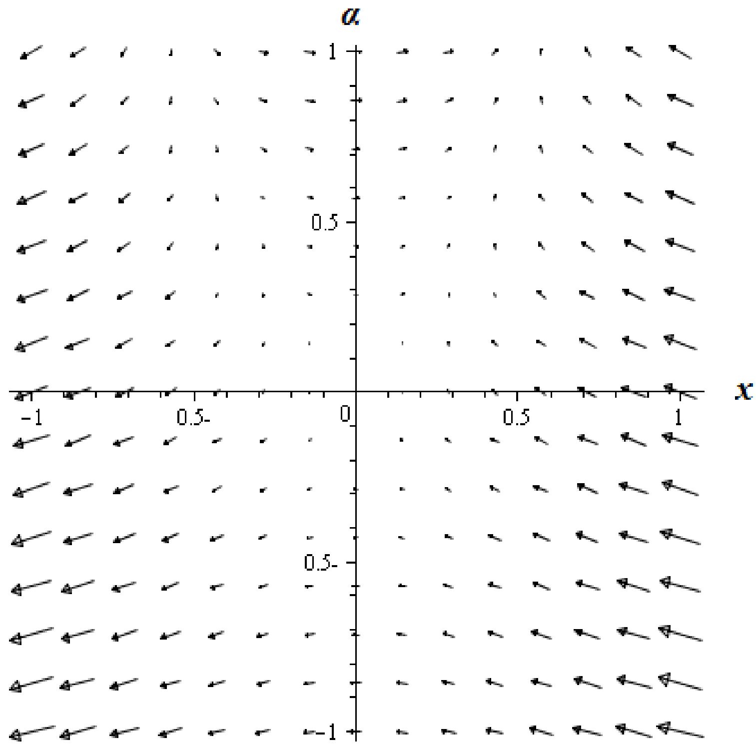

Figure-4: The vector field direction for the normal form of transcritical bifurcation.

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