Journal homepage www.jzs.univsul.edu.iq
Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

Hollow and Semihollow Modules

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Article info

Original: 24 August 2016
Revised: 31 January 2017
Accepted: 22 February 2017
Published online: 20 June 2017

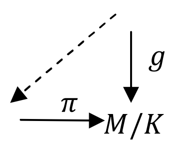
Key Words: *Hollow module, Semihollow Module, Small Submodules.*

Abstract

Let R be an associative ring with identity and M be a non-zero unitary left module over R. M is called a hollow (semihollow) module if every proper (finitely generated proper) submodule of M is a small submodule of M. The purpose of this work is, to give a comprehensive study of hollow modules and semihollow modules. Moreover, we study the class of modules with finite spanning dimension. We supply the details of the proofs for almost all the results and we illustrate the concepts by examples. Also, we add some results that seem to be new to the best of our knowledge.

Introduction

P. Fleury [3] in 1974, introduced the concept of hollow modules as follows: A module M is said to be hollow if $M \neq 0$ and whenever $M = A + B$, where A and B are submodules of M, then either $M = A$ or $M = B$. That is, M is hollow if $M \neq 0$ and every proper submodule of M is a small submodule of M. Hollow modules are dual to uniform modules. In fact, this type of modules is called couniform modules[1]. In 1977, K.M. Rangaswamy [14] generalized the concept of hollow modules to semihollow modules where a module M is called a semihollow module if $M \neq 0$ and every proper finitely generated submodule of M is a small submodule of M. Also it studies two classes of modules and the modules with finite spanning dimension which are defined by P.Fleury [4]. Let us recall that a module M is called self- projective module, if for any homomorphism $g: M \to M/K$, where K is a submodule of M there exists a homomorphism $h: M \to M$ such that $\pi \circ h = g$, where $\pi : M \to M/K$ is the natural epimorphism. That is,the following diagram is commutative:



We investigate some conditions under which certain hollow modules have a local endomorphism rings. And we discuss the concept of local modules which are generalization of local rings.

Finally, we remark that all rings are assumed to be associative with non-zero identity element and all modules are unitary left modules, unless otherwise stated.

Some Results on Small Submodules and the Jacobson Radical of a Module

In this section, we recall the definition of small submodules and we discuss some of the basic properties of this type of submodules. Furthermore, the concepts of the jacobson radical of a module is defined and characterized in term of small submodules and some of their properties are discussed.

Definition 1[9]: A submodule N of a module M is called a small submodule of M, denoted by $N \ll M$, if $N + L \neq M$, for any proper submodule L of M.

Remark 1: It is obvious from the definition that**,** a small submodule of a module M is always a proper submodule of M and if $M \neq 0$, then $\{0\} \ll M$.

Examples:

(1) Every proper submodule of the Z −module Z_{p^∞} is a small submodule of Z_{p^∞}.

(2) The submodule $\{\bar{0}, \bar{3}\}$ of the Z −module Z_6 is not a small submodule of Z_6.

Proposition 1: Let $K \le N \le M, K' \le N' \le M$ and $N/K \ll M/K$, $N'/K' \ll M/K'$, then $(N + N')/(K + K') \ll M/(K + K')$.

Proof: Define $f_1 : M/K \to M/(K + K')$ by $f_1(x + K) = x + (K + K')$, $x \in M$. Clearly, f_1 is well-defined and a homomorphism. But $N/K \ll M/K$, then $f_1(N/K) \ll M/(K + K')$. that is,

$$(N + K')/(K + K') \ll M/(K + K') \quad (1)$$

Also, define $f_2 : M/K' \to M/(K + K')$ by $f_2(m + K') = m + (K + K')$, f_2 is a well-defined and homomorphism. Since $N'/K' \ll M/K'$, then

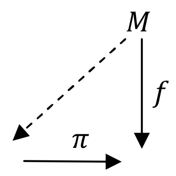
$$f_2(N'/K') = (N' + K)/(K + K') \ll M/(K + K') \quad (2)$$

From (1) and (2), we have $(N + N')/(K + K') = (N + K')/(K + K') + (N' + K)/(K + K') \ll M/(K + K')$.

The next proposition shows that if N is a submodule of a self - projective module M with $Hom(M, N) = 0$, then $N \ll M$.

Proposition 2: Let M be a self-projective module and let N be a submodule of M with $Hom(M, N) = 0$, then $N \ll M$.

Proof: Suppose that N is not a small submodule of M, so there exists a proper submodule K of M such that $M = N + K$. Consider the following diagram:



Where $\pi : M \to M/(K \cap N)$ is the natural epimorphism and $f : M \to M/(K \cap N)$ is define as follows:

For $m \in M$ and since $M = N + K$, thus there exists $n \in N$ and $k \in K$ such that $m = n + k$. Now, set $f(m) = n + (K \cap N)$, we claim that f is a well- defined, for if $n + k = n_1 + k_1$, then $n - n_1 = k - k_1$ which implies that $n - n_1 = k - k_1 \in K \cap N$ and hence $n + (K \cap N) = n_1 + (K \cap N)$. Also $f \neq 0$ since otherwise $N \le K$ and hence $M = N + K = K$ which is a contradiction. Since M is a self -projective module then there exists $\psi : M \to M$ such that $\pi \circ \psi = f$. Now, for $m \in M$ we have $(\pi \circ \psi)(m) = f(m)$.

$\psi(m) + K \cap N = n + K \cap N$, where $m = n + k$ for some $n \in N$ and $k \in K$. $\psi(m) - n \in K \cap N \leq N$ which implies that $\psi(m) \in N$. Thus $\psi(M) \leq N$, but $Hom(M, N) = 0$. That is, $\psi = 0$, which is a contradiction, since $\pi \circ \psi = f$ and $f \neq 0$. Hence $N \ll M$.

Definition 2: [6] Let M be a module, if there exist maximal submodules of M, then the intersection of all maximal submodules of M is called the Jacobson radial of M and denoted by $Rad\ M$. If there is no maximal submodule of M, then we define $Rad\ M = M$.

The following result is a generalization of the fact that every finitely generated submodule of Q is a small submodule of Q.

Proposition 3: Let M be a module, then $Rad\ M = M$ if and only if every finitely generated submodule of M is a small submodule of M.

Proof: Suppose that $Rad\ M = M$ and let N be a finitely generated submodule of M with $N + L = M$, hence $N = Rx_1 + Rx_2 + \cdots + Rx_n$ where $x_i \in M = RadM$ implies $Rx_i \ll M$, therefore $L = M$.

Conversely, let $m \in M$ implies $< m >= Rm$ is finitely generated implies that $< m >\ll M$ and hence $\langle m \rangle \leq RadM$, thus $m \in Rad\ M$.

Proposition 4: Let M be a module, if $RadM \ll M$, then $M/Rad\ M$ has no non-zero small submodule.

Proof: Let $L/RadM$ be a small submodule of $M/RadM$, then $L \ll M$. To see this, let $L + K = M$, then $(L/RadM) + ((K + RadM)/RadM) = M/RadM$, implies that $K + RadM = M$ and hence $K = M$.Therefore $L \leq RadM$, That is, $L = RadM$. Thus $L/RadM$ is a zero submodule of M.

Supplemented Modules and Coclosed Submodules

We recall in this section the following concepts: Supplemented, f −supplemented, amply supplemented and weakly supplemented modules and we investigate some of their properties, which are needed later in our work. Also the concept of coclosed submodule is discussed in their relationship with supplements.

Definition 3 [16]: A submodule V of a module M is called a supplement of U, where U is a submodule of M, if V is a minimal element in the set of submodules $L \leq M$ with $U + L = M$.

Remark 3: Supplements of submodules may not exist, for example $2Z$ as a submodule of Z has no supplement in Z, since the only small submodule of Z is 0.

Example: Let $U = \{\bar{0}, \bar{2}\} \leq Z_4$,then Z_4 is a supplement of U in Z_4.

Remark 4: (1) The above example shows that, if V is a supplement of U in M, then U need not be a supplement of V in M.

(2) Every direct summand has a supplement.

Definition 4 [16,18]: Let M be a module, then:

(1) M is called a supplemented module if every submodule of M has a supplement in M.

(2) M is called an $f-$ supplemented module if every finitely generated submodule of M has a supplement in M.

(3) M is called amply supplemented module, if for any two submodules U and V of M with $U + V = M$, V contains a supplement of U in M.

(4) M is called a weakly supplemented module, if for each submodule U of M, there exists a submodule V of M such that $M = U + V$ and $U \cap V \ll M$.

We have the following implications.

Remark 5[1]: Amply supplemented → Supplemented → Weakly supplemented.

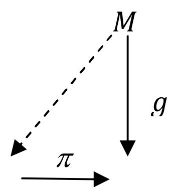
Next we give more properties of supplemented modules.

Proposition 5: Epimorphic image of a supplemented module is supplemented.

Proof: Let $f: M \to M'$ be an epimorphism with M supplemented. We must show that M' is supplemented. Let $K \leq M'$, then $f^{-1}(K) \leq M$, but M is supplemented so, there is a supplement L of $f^{-1}(K)$ in M. That is, $f^{-1}(K) + L = M$ and $f^{-1}(K) \cap L \ll L$. Now, $f\,(f^{-1}(K) + L)) = f\,(M)$ and hence $f\,(f^{-1}(K)) + f\,(L) = f\,(M)$. Therefore $K + f\,(L) = M'$. We need only to show that $K \cap f\,(L) \ll f\,(L)$, since $f^{-1}(K) \cap L \ll L$ then $f\,(f^{-1}(K) \cap L) \ll f\,(L)$. But one can easily show that $K \cap f\,(L) = f\,(f^{-1}(K) \cap L)$. Thus, $K \cap f\,(L) \ll\ f\,(L)$ and hence $f\,(L)$ is a supplement of K in M'.

Proposition 6: Let M be a supplemented self -projective module, then M is an amply supplemented.

Proof: Suppose that $M = U + V$, since M is supplemented, there exists a supplement X of U in M. That is, $M = U + X$ with $U \cap X \ll X$. Now, consider the following diagram:



Where $\pi : M \to M/U \cap V$ is the natural epimorphism, and $g : M \to M/U \cap V$ defined by $g(x) = v + U \cap V$, where $x = u + v$, for some $u \in U$ and $v \in V$. It can be easily shown that g is well- defined and homomorphism. Since M is a self - projective module, there exists a homomorphism $\psi : M \to M$. such that $\pi \circ \psi = g$. We claim that $\psi\,(M) \leq V$. To see this, let $m \in M$, then $(\pi \circ \psi)\,(m) = g\,(m)$, this implies $\psi\,(m) + U \cap V = v + U \cap V$ where $m = u + v$ for some $u \in U$ and $v \in V$. Therefore, $\psi\,(m) - v \in U \cap V \leq V$, that is, $\psi\,(m) \in V$ for all $m \in M$. Thus $\psi\,(M) \leq V$. Now, we show that $(I - \psi)\,(M) \leq U$. Let $m \in M$, then $m - \psi\,(m) = u + v - \psi\,(m)$ and by the last argument $v - \psi\,(m) \in U \cap V \leq U$. Thus $m - \psi\,(m) \in U$, for all $m \in M$ and therefore, $(I - \psi)\,(M) \leq U$. Also for $u \in U$ we have $u - \psi\,(u) \in U$ and hence, $\psi\,(u) \in U$, then $\psi\,(U) \leq U$. We can show that $\psi(X)$ is a supplement of U in V. Let $m \in M$, then $m = u + x$, for some $u \in U$ and $x \in X$. But $m - \psi\,(m) \in U$, thus $m - \psi\,(m) = u_1$, for some $u_1 \in U$ and hence, $m = \psi\,(m) + u_1 = \psi(u + x) + u_1 = \psi\,(u) + \psi\,(x) + u_1 = \psi\,(u) + u_1 + \psi\,(x) \in U + \psi\,(X)$. Therefore, $M = U + \psi(X)$. Now, $U \cap X \ll X$ and hence, $\psi\,(U \cap X) \ll \psi\,(X)$. (3)

But $U \cap \psi\,(X) \leq \psi(U \cap X)$, since if $w \in U \cap \psi\,(X)$, then $w = \psi\,(x) \in U$ for some $x \in X$. Also, $x - \psi\,(x) \in U$, thus $x - w \in U$ and hence, $x \in U$, which implies $w \in \psi\,(U \cap X)$. Now, if $y \in \psi\,(U \cap X)$ implies $y = \psi\,(t), t \in U \cap X$, then $\psi\,(t) \in \psi\,(U) \leq U$ and $\psi\,(t) \in \psi\,(X)$. Thus, $\psi\,(t) \in U \cap \psi\,(X)$. we get

$\psi\,(U \cap X) = U \cap \psi\,(X)$. (4)

Also by above the argument $\psi(X) \leq \psi\,(M) \leq V$ from (3) and (4), we get $U \cap \psi\,(X) \ll \psi\,(X)$. Thus, $\psi\,(X)$ is a supplement of U in V. That is, M is amply supplemented.

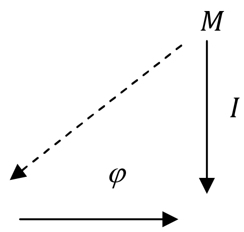
It is known that if U is a supplement of V ina module M and V is a supplement of U in M, then U and V are called mutual supplements.

Proposition 7: Let M be a supplemented and projective module, then M is amply supplemented and the intersection of mutual supplements is zero.

Proof: Since M is projective, then it is self - projective, therefore M is amply supplemented. Let U and V be mutual supplements submodules of M. Then $M = U + V$, $U \cap V \ll U$ and $U \cap V$, We have $(U \cap V) \oplus (U \cap V) \ll U \oplus V$. Let $K = \{ (x, -x) : x \in U \cap V\}$, implies that $K \leq (U \cap V) \oplus (U \cap V)$. Define $\varphi : U \oplus V \to M$ by $\varphi\,(u, v) = u + v$, for all $u \in U$ and $v \in V$.

Clearly φ is an epmimorphism.

Consider the following diagram:



Where $I : M \to M$ is the identity homomorphism. Since M is projective, so there exists a homomorphism h: $M \to U \oplus V$ such that $\varphi \circ h = I$. This implies that $\ker \varphi$ is a direct summand of $U \oplus V$.

But $\ker \varphi = K$ and $K \leq (U \cap V) \oplus (U \cap V) \ll U \oplus V$. Therefore, $K \ll U \oplus V$. Since K is a direct summand of $U \oplus V$, then $U \oplus V = \ker \varphi \oplus L$, for some submodule L of $U \oplus V$. That is, $\ker \varphi = 0$. It is enough to show that $U \cap V = 0$, to see this: let $w \in U \cap V$, then $\varphi\,(w, -w) = w + (-w) = 0$, That is,$(w, -w) \in \ker \varphi = (0, 0)$, thus $w = 0$.

Definition 5 [5]: A submodule N of a module M is called coclosed in M if $N/K \ll M/K$ implies $N = K$, for all submodule K of M contained in N.

Examples:

(1) $\{\bar{0}, \bar{2}, \bar{4}\}$ is a coclosed submodule in the Z −module Z_6. In fact, every direct summand of M is coclosed in M.

Proof: Let N be a direct summand of M, then there is a submodule L of M such that $M = N \oplus L$. Let K be a submodule of N and suppose that $N/K \ll M/K$. Now, $M/K = (N + L)/K = (N/K) + (L + K)/K$, this implies that $L + K = M$ and hence, $N = M \cap N = (L + K) \cap N = L \cap N + K = K$.

(2) The submodule $2Z$ of the $Z-$ module Z is not coclosed submodule in , since $2Z/4Z \ll Z/4Z$, but $2Z \neq 4Z$.

Hollow Modules

In this section, we recall the definition of hollow modules and we study the basic properties of this type of module. Also we add some new concepts with their proofs and investigate some conditions under which certain hollow modules have local endomorphs rings.

Definition 6 [3]: A non-zero module M is called a hollow module if every proper submodule of M is a small submodule of M.

Examples:

(1) Every simple module is a hollow module.

(2) Z_{p^∞} as a Z −module is a hollow module. In fact every uniserial module is hollow, where by a uniserial module we mean a module in which its submodules are linearly ordered by inclusion.

(4) Q as a Z −module is not hollow, since $Q = Q_p + Q^p$, where p is prime and Q_p is the submodule of Q consisting all rational numbers whose denominators are relatively prime to p and Q^p is the submodule of Q consisting all rational numbers whose denominators are powers of p. Also observe that $Q_p \cap Q^p = Z$.

Proposition 8: Epimorphic image of a hollow module is hollow.

Proof: Let M be a hollow module and $f: M \to M'$ an epimorphism with M' is a module. Suppose N' is a proper submodule of M' with $N' + K' = M'$, where $K' \leq M'$. Now, $f^{-1}(N')$ is a proper submodule of M since otherwise $f^{-1}(N') = M$ and hence $f\left(f^{-1}(N')\right) = f(M) = M'$, implies that $N' = M'$, which is a contradiction. Thus, $f^{-1}(N')$ is a proper submodule of M and therefore, $f^{-1}(N') \ll M$ and hence $f\left(f^{-1}(N')\right) \ll f(M)$, that is, $N' \ll M'$.

The following results proves another property of hollow modules.

Proposition 9: Let K be a small submodule of a module M. If M/K is a hollow module, then M is hollow.

Proof: Suppose that M/K is a hollow module with $K \ll M$. Let N be a proper submodule of M with $M = N + L$, where $L \leq M$. Then $M/K = (N + L)/K$, implies that $M/K = ((N + K)/K) + ((L + K)/K)$, since, $(N + K)/K$ is a proper submodule of M/K, then $(N + K)/K \ll M/K$ and hence, $(L + K)/K = M/K$. Therefore, $L + K = M$, but $K \ll M$, then $L = M$, that is, M is a hollow module.

Corollary 1: Let M be a hollow module, then M/N is also a hollow module, for every proper submodule N of M.

Proof: Let N be a proper submodule of M and $\pi: M \to M/N$ be the natural epimorphism, then M/N is a hollow module.

The following proposition can be easily verified.

Proposition 10: Every hollow module is amply supplemented.

Proof: Let M be a hollow module and let U be a proper submodule of M. Since M is a hollow module, then $U + M = M$ and $U \cap M = U \ll M$.

Proposition 11: Every non-zero coclosed submodule of a hollow module is hollow.

Proof: Let M be a hollow module and let N be a non-zero coclosed submodule of M. Suppose L is a proper submodule of N, then L is a proper submodule of M. Therefore $L \ll M$ and hence $L \ll N$.Thus N is a hollow module.

Another property of hollow modules is given below.

Proposition 12: Let M be a module, then M is a finitely generated hollow module if and only if M is a cyclic and has a unique maximal submodule.

Proof: Let M be a finitely generated hollow module, then $M = Rx_1 + Rx_2 + \cdots + Rx_n$ for $x_i \in M$ and $i = 1, 2, \ldots, n$. If $M \neq Rx_1$, then Rx_1 is a proper submodule of M, which implies that $Rx_1 \ll M$. Hence, $M = Rx_2 + Rx_3 + \cdots + Rx_n$. Sorepeat this argument finitely many times until we get $M = Rx_i$ for some i. Thus M is a cyclic module. Suppose M_1 and M_2 are two distinct maximal submodules in M, then $M = M_1 + M_2$, since M is a hollow module, then $M = M_1$ or $M = M_2$ which is a contradiction. *Conversely*, let M be a cyclic module having a unique maximal submodule, say N, then M is finitely generated. Let L be a proper submodule of M with $L + K = M$, where K is a submodule of M. Now, if $K \neq M$, then K is a proper submodule of M, and hence K is contained in a maximal submodule, since M finitely generated. But by assumption M has a unique maximal submodule N, thus L is contained in N. Therefore, $L + N = N = M$, which is a contradiction. Hence, $K = M$, thus, $L \ll M$. That is, M is a hollow module.

The relation between hollow module and indecomposable module is given in the following proposition.

Proposition 13: Every hollow module is indecomposable.

Proof: Suppose M is decomposable, then there are proper submodules K and L such that $M = K \oplus L$. But M is a hollow module, then either $K = M$ or $L = M$, which is a contradiction.

We recall that if M is finitely generated, then M/N is a finitely generated for every submodule N of M. But the converse is not true. The following proposition shows if M is a hollow module and M/N is finitely generated then M is also finitely generated.

Proposition 14: Let N be a proper submodule of a module M. If M is a hollow module and M/N is finitely generated, then M is finitely generated.

Proof: Let N be a proper submodule of a hollow module M with M/N is finitely generated. Then, $M/N = R(x_1 + N) + R(x_2 + N) + \cdots + (x_n + N)$, where $x_i \in M$ for all $i = 1, \ldots, n$. We claim that $M = Rx_1 + Rx_2 + \cdots + Rx_n$. Let $m \in M$ then $m + N \in M/N$, that is, $m + N = r_1(x_1 + N) + r_2(x_2 + N) + \cdots + r_n(x_n + N) = r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$, this implies that $m = r_1x1 + r_2x2 + \cdots + r_nx_n + n'$, for some $n' \in N$. Thus, $M = r_1x_1 + r_2x_2 + \cdots + r_nx_n + N$ and since M is a hollow module, then $N \ll M$, which implies that $M = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. That is, M is finitely generated.

We need the following lemma later.

Lemma 1: Let M be a hollow module, which has a maximal submodule K, then $RadM = K$.

Proof: Let L be another maximal submodule in M, then $K + L = M$. But M is a hollow module, thus $K = M$, which is a contradiction. Hence $RadM = K$.

Definition 7 [13]: Let M be a module, M is called a lifting module (or satisfieds $D1$), if for every submodule N of M there are submdules K and K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll M$.

Proposition 14[7 , 1-1-7]: Every hollow module is lifting.

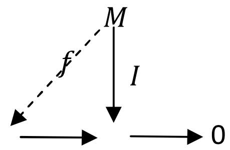
Proposition 15: Every indecomposable lifting module is hollow.

Proof: Let N be a proper submodule of a lifting module M, then $M = K\oplus K'$, $K \leq N$ and $N \cap K \ll K$. But M is indecomposable, thus $K' = 0$ and hence, $K = M$, which implies that, $N \cap M = N \ll M$.

Definition 8 [1]: A pair (P, f) is a projective cover of the module M in case P is a projective module and $f: P \to M$, where f is an epimorphism and $\ker f \ll P$. We also employ natural variations and abbreviations of this terminology; for example, we may call P itself a projective cover of M.

Proposition 16: Let M be a self-projective module, then M is a hollow module if and only if $End\ (M)$ is a local ring.

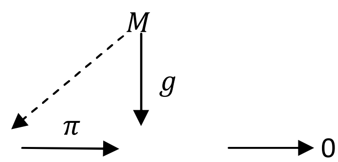
Proof: Let M be a hollow module and $0 \neq f \in End\ (M)$, if f is onto then since M is self-projective, there exists a homomorphism $g: M \to M$ such that $f \circ g = I$, where $I: M \to M$ is the identity homomorphism. That is, the following diagram is commutative:



$f \circ g = I$

Which implies that f has a right inverse and hence $M = \ker f \oplus g(M)$. Since M is a hollow module, then either $\ker f = M$ or $g\ (M) = M$. $\ker f \neq M$, since otherwise $f(M) = 0$ and hence, $f = 0$, which is a contradiction.Thus, $g\ (M\ ) = M$, that is, $\ker f = 0$, which implies that f is one to one. If f is not onto, then since $M = f\ (\ M\ ) + (I - f\ )(\ M)$ and M is hollow, we have $(I - f)$ is onto and above argument $(I - f)$ is one to one.

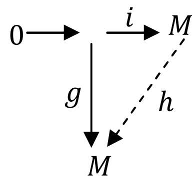
Conversely, suppose that $End\ (M)$ is a local ring. Let L be a proper submodule of M with $L + K = M$ for some submodule K of M. We have, $M/(\ L \cap K) = (L + K)/(\ L \cap K)$. Let $g: M \to M/(\ L \cap K)$ defined by $g\ (m) = k + L \cap K$, g is well-defined to see this, let $l + k = l_1 + k_1$, then $l - l_1 = k_1 - k \in L \cap K$. That is, $k + L \cap K = k_1 + L \cap K$ and it is clear g is a homomorphism. Since M is self-projective, then there exists a homomorphism $\alpha: M \to M$ such that the following diagram is commutative:



$\pi \circ \alpha = g$

Where $\pi: M \to M/(L \cap K)$ is the natural epimorphism. We have to show $\alpha\ (M\ ) \leq K$ and $K = \alpha(\ M) + L \cap K$. Let $m \in M$ then $\pi \circ \alpha\ (m)\ = g\ (m)$, hence $\alpha\ (m) + L \cap K = k + L \cap K$. Therefore, $\alpha\ (m) - k \in L \cap K \leq K$, that is, $\alpha\ (m) \in K$, thus $\alpha\ (M) \leq\ K$. Let $x \in K$, then $x = \alpha(x) - x + \alpha(x)$, $\alpha\ (x) \in \alpha\ (M)$ and $\pi\ (x - \alpha\ (x)) = \pi\ (x) - \pi \circ \alpha\ (x) = x + L \cap K - \alpha(x) + L \cap K = L \cap K$, then $x - \alpha\ (x) \in \ker \pi$. But $\ker \pi = L \cap K$, so that, $K = \alpha\ (M) + L \cap K$. Now, since $M = L + K$, then $M = L + \alpha\ (M) + L \cap K$, but $L \cap K \leq L$, therefore, $M = L + \alpha\ (M)$.Thus, $\alpha\ (M)$ is not a small submodule of M, since L is proper. Hence, $\alpha \notin\ Rad\ End\ (M)$, that is, α is an isomorphism. Finally, we obtain $= M + L \cap K = M$, so that, M is a hollow module.

We recall that a module M is called self-injective if for any homomorphism $g: K \to M$, where K is a submodule of M, there exists a homomorphism $h: M \to M$ such that $h \circ i = g$,where $i: K \to M$ is the inclusion homomorphism. That is, the following diagram is commutative



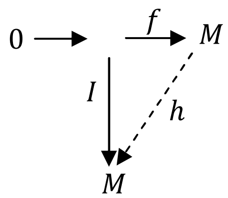
$h \circ i =$

Proposition 17: Let M be a self-injective hollow module, then $End\ (M)$ is a local ring.

Proof: Let M be a self-injective hollow module and let $0 \neq f \in End\ (M)$. Now, $\ker f \cap \ker(I - f) = 0$ and to see this, let $w \in \ker f \cap \ker(I - f)$, then $f(w) = 0$. Also, $(I - f)\ (w) = w - f\ (w) = 0$, this implies that $w = 0$. Since M is hollow, then M is indecomposable and hence by the previous lemma either $\ker f = 0$ or $\ker(I - f) = 0$, that is, either f or $I - f$ is a monomorphism.

Case 1: If f is a monomorphism.

Consider the following diagram:



Where $I: M \to M$ is the identity homomorphism. Since M is self-injective, there exists $h: M \to M$ such that h∘f=I and this implies that f has a left inverse. That is, $M = f\ (M) \oplus \ker h$, but M is indecomposable and $f\ (M) \neq 0$. Thus $M = f\ (M)$.That is, f is an epimorphism and hence f is invertible.

Case 2: If $(I - f\ )$ is a monomorphism.

The proof is similar to case 1.Where $\pi: M \to M/(L \cap K)$ is the natural epimorphism. We have to show $\alpha\ (M\ ) \leq K$ and $K = \alpha(\ M) + L \cap K$. Let $m \in M$, then $\pi \circ \alpha\ (m) = g\ (m)$, hence, $\alpha\ (m) + L \cap K = k + L \cap K$. Therefore, $\alpha\ (m) - k \in L \cap K \leq K$, that is, $\alpha\ (m) \in K$, thus $\alpha\ (M) \leq K$. Let $x \in K$, then $x = \alpha\ (x) - x + a(x), \alpha\ (x) \in \alpha\ (M)$ and $\pi\ (x - \alpha\ (x)) = \pi\ (x) - \pi \circ \alpha\ (x) = x + L \cap K - \alpha\ (x) + L \cap K = L \cap K$, then $x - \alpha\ (x) \in \ker \pi$ but $\ker \pi = L \cap K$, That is, $K = \alpha\ (M) + L \cap K$. Now, since $M = L + K$, then $M = L + \alpha\ (M) + L \cap K$, but $L \cap K \leq L$. Therefore, $M = L + \alpha\ (M)$.Thus, $\alpha\ (M)$ is not a small submodule of M, since L is proper. Hence, $\alpha \in Rad\ End\ (M)$, That is, α is an isomorphism. Finally, we obtain $K = M + L \cap K = M$, That is, M is a hollow module.

Local Modules

In this section, we recall the definition of local modules and we give some properties of this class of modules. Also we present conditions under which a module has a local submodule.

Definition 9[8]: Let M be a module, M is called a local module if M has a unique maximal submodule N, which contains all proper submodules of M.

Examples:

(1) The Z −module Z_4 is a local module.

(2) Q as a Z −module is not a local module.
(3) $Z_2 \oplus Q$ has a unique maximal submodule, but does not contain all proper submodules, since $Z_2 \oplus \{0\}$ is a proper submodule of $Z_2 \oplus Q$, which is not contained in the uniqe maximal submodule $\{0\} \oplus Q$ of the module $Z_2 \oplus Q$.

A characterization of local modules is given in the following proposition.

Proposition 18: Let M be a module, M is a local module if and only if M is hollow module and has a unique maximal submodule.

Proof: Suppose that M is a local module, then M is a hollow module, also M has a unique maximal submodule. Conversely, let M be a hollow module, which has a unique maximal submodule, say N, we only have to show that M is a cyclic module. Let $w \in M - N$, then, $Rw + N = M$ and since M is a hollow module, then N is a small submodule of M and hence $M = Rw$. Therefore, M is a cyclic module.

Examples:

(1) Z_{p^∞} is a hollow module but not local.
(2) Z_6 is a cyclic module but not local.

The following proposition gives another characterization of local modules.

Proposition 19:
Let M be a module, M is a local module if and only if it is a cyclic module and every non-zero factor module of M is indecomposable.

Proof: Trivial by [11, proposition 1.3.11].

Proposition 20: Let M be a module, M is a local module if and only if $RadM$ is small and maximal in M.

Proof: Let $RadM$ be a small and maximal submodule in M. First we show that $RadM$ is a unique maximal submodule in M. Suppose L is another maximal submodule in M, then $M = L + RadM$, but $RadM \ll M$ which implies that $L = M$, which is a contradiction. Thus, $RadM$ is the unique maximal submodule in M. We claim every proper submodule of M is contained in $RadM$. Let N be a proper submodule of M, if N is not contained in $RadM$, then $N + RadM = M$. But $RadM \ll M$, which implies that $N = M$ then we have a contradiction. Therefore, M is a local module. Conversely, trivial by [11, proposition 1.3.13].

Before we state the next proposition, we need the following lemma.

Lemma 2: Let (P, h) be a projective cover for M, then M has a maximal submodule.

Proof: Since P is a projective module and $h: P \to M$ is epimorphism with $\ker h \ll P$, then P has a maximal submodule, say N, [1,17.14]. We claim that $h(N)$ is a maximal submodule in M. First $h(N) \neq M$, since otherwise $h(N) = h(P)$ and this implies that $P = N + \ker h$. To see this let $x \in P$ then $h\,(x) \in h(P) = h\,(N)$ and there exists $n \in N$, such that $h\,(x) = h\,(n)$, That is, $x - n \in \ker h$ and hence $x \in N + \ker h$. Thus, $P = N + \ker h$, but $\ker h \ll P$ and hence $N \neq P$, which is a contradiction, since $N \neq P$. Next we show that $M = h\,(N) + Rm$, for all $m \in M - h\,(N)$. Since h is onto, then there exists $x \in P$ such that $h\,(x) = m$ and $x \notin N$, which implies that $N + Rx = P$. Now, $h\,(N + Rx) = h\,(P) = M$, then $h\,(N) + R\,h(x) = M$. Hence, $h(N) + Rm = M$. That is, $h\,(N)$ is a maximal submodule of M.

The following proposition proves that supplements of maximal submodules of an module are local.

Proposition 21: Let K be a maximal submodule of a module M. If L is a supplement of K in M, then L is a local module.

Proof: Let L be a supplement of K and let L_1 be a proper submodule of L with $L_1 + L_2 = L$, for some submodule L_2 of L. Now, $K + L = M = K + L_1 + L_2 = M$ and $L_1 \leq K$ since otherwise $K + L_1 = M$ and by minimality of L, we get $L_1 = L$, which is a contradiction. Thus, $K + L_2 = M$ and again by minimality of L we get $L_2 = L$. That is, L is hollow. To show L is a cyclic module, let $x \in M - K$, then $Rx + K = M$ and this implies that $Rx = L$, by minimality of L. Thus, L is a local module.

Definition 10: [12] Let M be a non-zero module, then M is a prime module if $[0{:}\,M] = [0{:}\,N]$, for all non-zero submodule N of M, where $[0 : M] = \{r \in R{:}\, rM = 0\}$.

Lemma 3: Let M be a module, if M is a prime module, then $[0{:}\,M]$ is a prime ideal.

Proof: First $[0:M] \neq R$ since otherwise $1 \in [0:M]$, which implies $1M = 0$, That is, $M = 0$, which is a contradiction. Let $r_1 r_2 \in [0:M]$, if $r_2 \notin [0:M]$, then there exist $0 \neq m \in M$, such that $r_2 m \neq 0$, then $r_1(r_2 m) = (r_1 r_2)m = 0$, this implies $r_1 \in [0: < r_2 m >]$, but $[0: < r_2 m >] = [0:M]$ thus $[0:M]$ is a prime ideal.

Semihollow Modules

We give some basic properties of semihollow modules and some characterizations of semihollow modules in addition to some other results.

Definition 11 [14]: A non-zero module M is called semihollow module if every proper finitely generated submodule of M is a small submodule of M.

Remark 5: In [19], it is proved that, if $0 \neq M$ is a submodule with Rad$M = M$, then M is a semihollow module . Thus the Z −modules Q , Z_{p^∞} and $Q + Z_{p^\infty}$ are semihollow modules.

Remarks 6: Every hollow module is a semihollow module.

The converse of this remark is not true, for example Q as a Z −module is semihollow but not hollow since $Q = Q_p + Q^p$, for every prime p, where Q_p is the submodule of Q consisting all rational numbers whose denominators are relatively prime to p and Q^p is the submodule of Q consisting of all rational numbers whose denominators are powers of the prime p ,observe that $Q_p \cap Q^p = Z$.

The following are some basic properties of semihollow modules.

Proposition 22: Let M be a semihollow module, then M/N is a semihollow module, for all proper submodules N of .

Proof: Let U/N be a proper finitely generated submodule of M/N with $U/N + V/N = M/N$ for some submodule V of M, then $U/N = R(x_1 + N) + R(x_2 + N) + \cdots + R(x_n + N) = Rx_1 + Rx_2 + \cdots + Rx_n + N$. Now, $M/N = (Rx_1 + Rx_2 + \cdots + Rx_n)/N + V/N = (Rx_1 + Rx_2 + \cdots + Rx_n + V)/N$. Then, $Rx_1 + Rx_2 + \cdots + Rx_n + V = M$, but M is a semihollow module, then each $Rx_i \ll M$, that is $V = M$.

Corollary 2: Let M be a semihollow module and $h: M \to M'$ an epimorphism, then M' is a semihollow module, where M' is a non-zero module.

Proof: Since M is a semihollow module and $h: M \to M'$ is an epimorphism ,then $M/\ker h$ is a semihollow module and by first isomorphism theorem $M/\ker h \cong M'$. Hence M' is a semihollow module.

Next, we give a condition under which the converse of proposition 22 is true.

Proposition 23: Let $N \ll M$, if M/N semihollow module, then M is a semihollow module.

Proof: Let K be a proper finitely generated submodule of M with $K + L = M$, then $M/N = (K + L)/N = (K + N)/N + (L + N)/N$. Since K is a finitely generated submodule, then $K = Rx_1 + Rx_2 + \cdots + Rx_n$ for $xi \in M$ and $i = 1, \ldots, n$ and hence $(K + N)/N = (Rx_1 + Rx_2 + \cdots + Rx_n + N)/N = R(x1 + N) + R(x2 + N) + \cdots + R(xn + N)$. Thus, $(K + N)/N$ is finitely generated submodule of $/N$. That is, $(L + N)/N = M/N$ which implies that $L + N = M$,but N is small in M, therefore $L = M$.

Remark 7: in general, a submodule of a semihollow module need not be semihollow for example, Z is a submodule of the semihollow module Q, but Z is not semihollow since $2Z$ is a finitely generated submodule of Z, which is not a small submodule of Z.

Here we prove that every finitely generated submodules of semihollow modules has a supplement.

Proposition 24: Every semihollow module is f −supplemented.

Proof: Let M be a semihollow module and U a finitely generated submodule. Now, $M = M + U$ and $U \cap M << M$ .Thus, M is a supplement of U in M.

Proposition 25: Let M be a module and L a Noetherian semihollow submodule of M, then either $L \ll M$ or L is a coclosed submodule in M.

Proof: Suppose that L is not a coclosed submodule in M, then there exists a proper submodule K of L such that $L/K \ll M/K$. But L is a semihollow module, so every submodule of L is finitely generated hence, $K \ll L$ thus, $K \ll M$ [19]. Now, $K \ll M$ and $L/K \ll M/K$, then $L \ll M$.

Next we give a property of semihollow modules in the following proposition which proves that direct summands of semihollow modules are semihollow.

Proposition 26: The direct summand of a semihollow module is semihollow .

Proof: Let $M = M_1 + M_2$ be a semihollow module .Thus, $M/M_1 \cong M_2$ and M/M_2 is a semihollow. Thus, M_1 is semihollow .

Remark 8: The direct sum of semihollow modules is not semihollow in general.

For example, we know that Q and Z_p are semihollow modules, where p is a prime number, but $Q \oplus Z_p$ is not semihollow, since $Q \oplus Z_p$ has a maximal submodule $Q \oplus Z_p$ which is not local.

Next we restate the following propositions, the proofs of which can be found in[14].

Proposition 27 [14]: Let M be a module, then M is a semihollow module if and only if every proper cyclic submodule of M is a small submodule of M.

Proposition 28 [14]: Let M be a module, then M is a semihollow module if and only if M is a local module or has no maximal submodule.

Before, we give the next proposition we need the following lemma which can be found in [1,15.21].

Lemma 4: Let R be a left artinian ring .If M is a module, then Rad$M \ll M$, moreover for M the following statements are equivalent:

1- M is a finitely generated module.
2- M is a Noetherian module.
3- M is an Artinian module.

Proposition 29: Every semihollow module over an Artinian ring is Noetherian.

Proof: Let M be a semihollow module over an Artinian ring, then Rad$M \ll M$. That is, Rad$M \neq M$ and since M is a semihollow module, then M is a local module. Thus, M is a cyclic module, this implies that M is finitely generated and hence a Noetherian module.

Corollary 3: Every hollow module over an Artinian module is Noetherian.

In the following proposition we investigate some relationships between a module and it's radical in case the module is semihollow.

Proposition 30: Let M be a module, then:

1- If M is a semihollow module and Rad$M$ is a Noetherian module, then M is a Noetherian module.
2- If M is an artinian module and Rad$M$ is an Artinian module, then M is an Artinian module.

Proof:

1- Let M be semihollow module and Rad$M \neq M$, then M has a maximal submodule and hence M is a local module. Therefore, Rad$M$ is a maximal and a small submodule of M [19].Thus $M/$Rad$M$ is a simple module which implies that $M/$Rad$M$ is a Noetherian module . Now, $0 \to \text{Rad}M \to M \to M/\text{Rad}M \to 0$ is a short exact sequence and hence M is Noetherian.

Proposition 31: Let R be a field, then the simple modules over R are the only semihollow modules.

Proof: Clearly every simple module is a semihollow module .Let M be a semihollow module over a field R ,we have to show that M is a simple module. Since a module over a field is a vector space, then Rad$M \neq M$, thus M is a local module. Therefore, M is a cyclic module. That is $M = < x >$, $x \in M$. Let N be a submodule of M, if $N \neq 0$, then there exists $0 \neq w \in N$. Hence $w \in M$, this implies that $w = rx$ for some $0 \neq r \in R$. Since R is a field, then r^{-1} exists therefore, $x = r^{-1}w$ and hence $x \in N$. That is, $N = M$, thus M is a simple module.

Some Charactrizations of Semihollow Modules:

In this section, we recall the definitions of V −rings, perfect rings and max rings and we obtain some characterizations of semihollow modules over these types of rings. In section one, we have shown that M is a semihollow module if and only if M is a local module or has no maximal submodule.

In [14], The following result is proved:

Proposition 32 [14]: Let M be a module with Rad$M \neq M$, then M is a semihollow module if and only if M is a local module.

Proposition 33: Let M be a finitely generated module, then M is a semihollow module if and only if M is a local module.

Proof: Suppose M is a semihollow. Since M is finitely generated, then Rad$M \neq M$ and hence M is a local module. The prove of the converse part is trivial.

Corollary 4: Let M be a module, then M is a semihollow and a finitelygenerated module if and only if M is a cyclic module with the unique maximal submodule.

Proof: Let M be a semihollow and finitely generated, then M is local. Thus, M is hollow and we have M finitely generated, then M is cyclic and has a unique maximal submodule [19].

Conversely, let M be a cyclic module then it is finitely generated and hence every proper submodule of M is contained in a maximal submodule, but M has a unique maximal submodule, then M is local by definition Therefore M is asemihollow module.

Corollary 5: Let M be a non-zero module, then M is semihollow module and Rad$M \neq M$ if and only if M is semihollow and cyclic.

Proof: Let M be a semihollow and Rad$M \neq M$, then M is a local and hence M cyclic module.

Conversely, let M be a cyclic module, then M is a finitely generated module. Thus, M has a maximal submodule, which implies that Rad$M \neq M$.

The following proposition shows that a projective cover of a semihollow module is semihollow.

Proposition 34: Let (P, g) be a projective cover of M ,then M is a semihollow module if and only if P is a semihollow module.

Proof: Suppose that $g: P \to M$ is a projective cover of a semihollow module M. Since g is an epimorphism then $P/\ker g \cong M$ .Thus $P/\ker g$ is semihollow module ,but we have $\ker g \ll P$, therefore P is a semihollow module. Conversely , suppose that P is a semihollow module, where $g: P \to M$ is a projective cover of M, then M is a semihollow module.

Proposition 35: Let (P, g) be a projective cover of M, then M is a semihollow module if and only if M isa hollow module if and only if M is a local module.

Proof: Suppose that M is a semihollow module, then P is a semihollow module. Since P is a projective module, then Rad$P \neq P$ and hence, P local module, therefore P is a hollow module. Thus M is a hollow module [19]. Also, P is a local module, then by [19] M is a local module. The proof of the converse is trivial since every local module is a semihollow module.

Proposition 36: Let P be a projective module, then P is a semihollow module if and only if End(P) is a local ring.

Proof: Let P be a projective module then Rad$P \neq P$. Suppose P is a semihollow module, then P is a local module. Thus End(P) is a local ring[19].

Conversely, let P be a projective module and End(P) be a local ring, then P is a local module [19] and hence P is a semihollow module .

We recall that a ring R is called a left V −ring [2,P16] if Rad$M = 0$ for every left $R -$ module M.

The following proposition gives a condition under, which a semihollow module is hollow.

Proposition 37: Let M be a module over a V −ring, then M is a hollow module if and only if M is semihollow module.

Proof: Let M be a semihollow module over a V −ring R, then since Rad$M = 0 \neq M$, thus M is local and hence M is hollow . The proof of the converse part is trivial.

Also, one can show that over V −ring every semihollow module is simple.

Proposition 38: Let M be a moduleover a V −ring R, then M is a semihollow module if and only if M is a simple module.

Proof: Let M be a semihollow module over a V −ring R, then M is a local module and hence has a maximal submodule N. Therefore, Rad$M = N$, but R is a V −ring thus $N = 0$, That is M is a simple. Conversely is trivial.

Corollary 6: Let M be an R −module over a V −ring R, then the following are equivalent:

1- M is a hollow module.
2- M is a semihollow module.
3- M is a simple module.

We recall a ring R is called a left perfect ring if every module over R has a projective cover, [1,P3.15].

Proposition 39: Let M be an R −module over a perfect ring R, then the following are equivalent:

1- M is a hollow module.
2- M is a semihollow module.
3- M is a local module.

Proof: (1)⟹(2) : Trivial.

(2) ⟹ (3): Since R is a perfect ring, then M has a projective cover and hence M has a maximal submodule. Therefore, M is a local module.

(3) ⟹ (1) Trivial.

Recall that a ring R is called max ring [11] if every left module M over R has a maximal submodule.

Proposition 40: Let M be an R −module over a max ring R, then the following are equivalent:

1- M is a hollow module.
2- M is a semihollow module.
3- M is a local module.

Proof: (1) ⟹ (2) Trivial.

(3) ⟹ (3) Since R is a max ring, then M has a maximal submodule [19]. Hence Rad$M \neq M$. Therefore, M is a local module.

(3) ⟹ (1) Trivial.

Module with Finite Spanning Dimension

In this section, we recall the concept of modules with finite spanning dimensions. We give some basic properties of this class of modules. Also, we show that under certain conditions a module with finite spanning dimension is a finite direct sum of hollow modules.

Definition 12 [4]: Let M be a module, we say M has a finite spanning dimension if for every strictly decreasing sequence of submodules $U_0 \geq U_1 \dots$, there is i and $U_j \ll M$ for every $j \geq i$.

Example: Every hollow module has a finite spanning dimension.

Remark 9: Every Artinian module has a finite spanning dimension.

Proof: Let M be an artinian module and suppose M has no finite spanning dimension, there exists $U_0 \geq U_1 \geq \cdots$, such that there is no i with U_i is a small submodule of M. But M is an Artinian module, which is a contradiction. Thus, M has a finite spanning dimension.

The following proposition shows that a coclosed submodule of a module with finite spanning dimension has a finite spanning dimension.

Proposition 41: Let M be a module with finite spanning dimension, then every coclosed submodule of M has a finite spanning dimension.

Proof: Let K be a coclosed submodule of M. Suppose that $X_1 \geq X_2 \geq$ is a sequence of submodules of K, then $X_1 \geq X_2 \geq \cdots$ is a sequence of submodules of M, but M has a finite spanning dimension, then there exists i such that $X_j \ll M$ for all $j \geq i$. Now, $X_j \leq K \leq M$ and K is a coclosed submodule of M, then $X_j \ll K$ [11]. Thus K has a finite spanning dimension.

The following lemma gives a condition under which a module has a hollow submodule.

Lemma 5: Let $M = M_1 + M_2$, where M_1 and M_2 are proper submodules of a module M. If every proper submodule of M_1 is a small submodule of M, then M_1 is hollow.

Proof: Let K_1 be a proper submodule of M_i and suppose that $K_1 + L_1 = M_j$, then $M = K_1 + L_1 + M_2 = L_1 + M_2$ since $K_1 \ll M$. Now, $L_1 = M_1$ since otherwise $M = M_2$, which is a contradiction.

Proposition 42 [4]: Let M be a module with finite spanning dimension, then M has a hollow submodule.

Remark 10: In the proof of the previous proposition, we have shown that if M has a finite spanning dimension and M_1 is a submodule of M, which is not a small submodul of M, then M_1 contains a hollow submodule.

The next proposition shows that a finite spanning dimension module is amply supplemented.

Proposition 43 [4]: Let M be a module with finite spanning dimension, then M is an amply supplemented.

The following are some basic properties of a module with finite spanning dimension **[14].**

Proposition 44: Let $f: M \to M'$ be an epimorphism, where M and M' are modules. If M has a finite spanning dimension, then so is M'

Proof: Let $U_1' \geq U_2' \geq \cdots$ be a strictly decreasing sequence of submodules of M' Then $f^{-1}(U_1') \geq f^{-1}(U_2') \geq \ldots$ is a strictly decreasing sequence of submodules of M. But M has a finite spanning dimension, so there exists i such that $f^{-1}(U_j') \ll M$ for all $j \geq i$, and hence $U'j = f\left(f^{-1}\left(U_j'\right)\right) \ll M'$ for all $j \geq i$.

Corollary 7: Let N be a submodule of a finite spanning dimension module M, then M/N has a finite spanning dimension.

Proposition 45: Let M be a module with finite spanning dimension and S a submodule of M, which is not a small submodule of M, then M/S is an Artinian module.

Proof: Suppose that M/S is not an Artinian module, then there is an infinite strictly decreasing sequence of submodules $U_1/S \geq U_2/S \geq \ldots \geq U_n/S \geq \cdots$ of M/S, where $U_1 \geq U_2 \geq \ldots \geq U_n \geq \cdots$ is an infinite strictly decreasing sequence of submodules of M. But M has a finite spanning dimension, hence there is an integer i such that $U_j \ll M$ for all $j \geq i$. Since $S \leq U_j$ for all $j \geq i$, then $S \ll M$, which is a contradiction. Therefore, M is Artinian.

Proposition 46: Let M be a module with finite spanning dimension. If B is a supplement submodule of M, then B has a finite spanning dimension.

Proof: Let B be a supplement submodule of M, then by [11], B is a coclosed submodule of M and since M has a finite spanning dimension, therefore by [11], B has a finite spanning dimension.

Proposition 47: Let $M = M_1 \oplus M_2$, then M is a module with finite spanning dimension if and only if each of M_1 and M_2 has a finite spanning dimension.

Proof: Suppose $M = M_1 \oplus M_2$ has a finite spanning dimension since $M_1 = M/M_2$ and M/M_2 has a finite spanning. Thus M_i has a finite spanning dimension. Similarly one can do the same for M_2.

Conversely, suppose each of M_1 and M_2 has a finite spanning dimension. Let $K_1 \geq K_2 \geq \cdots$ be a strictly decreasing sequence of submodules of $M_1 \oplus M_2$. Let $f_i: M_1 \oplus M_2 \to M_1$ be the projection $i = 1, 2$, and $f_1(K_1) \geq f_1(K_2) \geq$ ... be a strictly decreasing sequence of submodules of M_1. Thus there exists i_2 such that $f_2\left(K_{j_2}\right) \ll M_2$ for all $j_2 \geq i_2$. Similarly, $f_2(K_i) > f_2(K_2) > \cdots$ is a strictly decreasing sequence of M_2 thus there exists i_2 such that $f_2\left(K_{j_2}\right) \ll M_2$ for all $j_2 > i_2$. Let $t = \max\{i_1, i_2\}$, therefore $f_i(K_s) \ll M$, and $f_2(K_s) \ll M_2$ for all $s > t, K_t \leq (f_1(K_t) \oplus f_2(K_t)) \ll M_1 \oplus M_2$. Thus $K_t \ll M$ for all $s > t$.

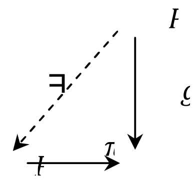
The following proposition gives a necessary and sufficient condition under which a projective module has a finite spanning dimension.

Proposition 48 [14]: Let P be a projective module, then P has a finite spanning dimension if and only if P is a local or an artinian module.

The next proposition gives a condition under which a self-projective module has a finite spanning dimension.

Proposition 49: Let P be a self-projective module, then P has a finite spanning dimension if and only if P is a hollow or an artinian module.

Proof: Let P be a self-projective module having a finite spanning dimension. Suppose P is not a hollow module, then there exist proper submodules $A_1, B_1$of P such that $P = A_1 + B_1$.We will show that $A \cap B = 0$. Consider the following diagram:



Where $\pi: P \to P/A \cap B$ is the natural epimorphism and g defined as follows, for $x \in P$, we have $x = a + b$ for some $a \in A$ and $b \in B$ such that $g(x) = a + A \cap B$. One can show that g is well-defined, since P is a self-projective module, then there exists $\alpha: P \to P$ such that $\pi \circ \alpha = g$. First, we have to show $\alpha(P) \leq A$. Let $x \in P$, then $(\pi \circ \alpha)(x) = g(x)$, That is, $\alpha(x) + A \cap B = a + A \cap B$, which implies that $\alpha(x) - a \in A \cap B \leq A$ and hence $\alpha(x) \in A$. Thus, $\alpha(P) \leq A$. Now, we must show $P = B + \alpha(P)$.

Let $m \in P, m = \alpha(m) - m + \alpha(m)$, where $m = a + b$ for some $a \in A$ and $b \in B$. $(\pi \circ \alpha)(m) = g(m)$ implies $\alpha(m) + A \cap B = a + A \cap B$, That is, $\alpha(m) - a \in A \cap B \leq B$,then $\alpha(m) - a \in B$ therefore, $\alpha(m) - a - b \in B$. That is, $\alpha(m) - (a + b) = \alpha(m) - m \in B$. Hence, $P = \alpha(P) + B$ but P has a finite spanning dimension then P is amply supplemented and hence supplemented, then $\alpha(P) = A$. We claim $P = A + \ker\alpha$. Notice that, $A = \alpha(P) = \alpha(A + B) = \alpha(A) + \alpha(B)$. We can show $\alpha(B) \leq A \cap B$ as follows:

Let $b \in B, \alpha(b) \in \alpha(B), (\pi \circ \alpha)(b) = g(b)$, then $\alpha(b) + A \cap B = A \cap B$, hence $\alpha(b) \in A \cap B$, thus $\alpha(B) \leq A \cap B$. Since, $A \cap B \ll A$, then $\alpha(B) \ll A$ [11]. Therefore, $A = \alpha(A) = \alpha(P)$ thus, $P = A + ker\,\alpha$. Now, we must show $\ker\alpha < B$. Let $y \in ker\,\alpha, y = \mathrm{a}_1 + \mathrm{b}_1$ for some $a_1 \in A$ and $b_1 \in B$. Now, $(\pi \circ \alpha)(y) = g(y)$, then $\alpha(y) + A \cap B = a_1 + A \cap B, 0 - a_1 \in A \cap B$, then $a_1 \in B$ and we have $b_1 \in B$, then $\mathrm{a}_1 + b_1 \in B$ implies $y \in B$. Thus, $\ker\alpha \leq B$. But B is a supplement of A, then $\ker\alpha = B$ hence, $\alpha(B) = 0$.

Finally, we must show that $\alpha(B) = A \cap B$, let $t \in B$. Then $t \in B = \alpha(P)$, there exists $s \in P$ such that $t = \alpha(s), (\pi \circ a)(s) = g(s)$, then $\alpha(s) + A \cap B = a + A \cap B$, where $s = a + b$, this implies $\alpha(s) - a \in A \cap B \leq B$. That is, $-a \in B$, but $a \in B$. Hence, $a + b = s \in B$ therefore $\alpha(s) = t \in \alpha(B)$. This implies that, $A \cap B \leq \alpha(B)$. Let $w \in \alpha(B)$, then $w = a(y)$ for some $y \in B$. Thus, $a(y) + A \cap B = 0 + A \cap B$ which implies that $w = a(y) \in A \cap B$ and hence $A \cap B = a(B) = 0$. Therefore, $P = A \oplus B$, $A \neq P \neq B$, then P/A and P/B are artinian, then B and A are artinian. Hence $P = A \oplus B$ is artinian.

Before we give the next proposition we need the following two lemmas.

Lemma 6: Let M be a module and let K be a coclosed submodule of M. If every proper submodule of K is a small submodule of M, then K is a hollow submodule of M.

Proof: Let X be a proper submodule of K with $X + Y = K$ for some submodule Y of K.

To show $Y = K$, it is enough to show that $K/Y \ll M/Y$. To verify this, let $K/Y + N/Y = M/Y$ for some submodule N/Y of M/Y with $Y \leq N$. Now, $(K+N)/Y = M/Y$, this implies that $K + N = X + Y + N = M$. Therefore, $X + N = M$ and hence $N = M$, since $X \ll M$. Thus, $K/Y \ll M/Y$.

Lemma 7: Let A, B and K be submodules of a module M such that $A \leq K \leq A + B$, then $K = A + K \cap B$.

Proof: Since $K = K \cap (A + B)$ and by modular law, we have $K = A + K \cap B$.

Now it is the time to prove the main result of this section.

Proposition 50 [4]: Let M be a module with finite spanning dimension then, there is an integer p and $M = H_1 + H_2 + \ldots + H_p$, where each H_j is a hollow submodule of M for $i = 1,2,\ldots,n$. Furthermore $H_1 + H_2 + \ldots + H_i + \ldots + H_p \neq M$. Finally, if $M = H'_1 + H'_2 + \ldots + H'_q$ satisfies the above two conditions, then $p = q$.

We recall the Krull-Schmidt-Azumaya Theorm which can be found in **[10, P.180].**

Krull-Schmidt-Azumoya

Theorem 1: Let $M = \oplus_{i \in I} M_i$, where $\text{End}(M_i)$ is local for all $i \in I$ and $M = \oplus N_j$, where N_j indecomposable and $N_j \neq 0$ for all $j \in J$, then a bijection $B: I \to J$ exists with $M_i \cong N_{p(i)}$ for all $i \in I$.

Proposition 51: [14] Let P be a self-projective module with finite spanning dimension, then P is a direct sum of a finite number of hollow modules $P = H_1 \oplus H_2 \oplus \ldots \oplus H_n$. If $\text{P} = H'_1 \oplus H'_2 \oplus \ldots \oplus H'_k$, where each H'_i is a hollow module, then $n = k$ and there is a bijection σ of $\{1, 2, \ldots, n\}$ to itself such that $H_i \cong H'_{\sigma(i)}, i = 1,2,\ldots,n$.

Proposition 52 [4]: Let M be a module with finite spanning dimension and $\text{Rad}M = 0$, then M is a finite direct sum of simple modules.

Corollary 8 [4]: Let M be a module, then $\text{Rad}\, M = 0$ and M has a finite spanning dimension if and only if $\text{Rad}\, M = 0$ and M is an artinian module.

Proposition 53: If M has a finite spanning dimension, then $M/\text{Rad}M$ is a direct sum of finitely many simple modules. Moreover, if $\text{Rad}M$ is not an essential submodule of M, then M is artinian.

Proof: Let M be a module with finite spanning dimension and $M^* = M/\text{Rad}M$. Now, M^* has a finite spanning dimension, but $\text{Rad}\, M^* = 0$ **[1,15.8]** and $M^*$is a finite direct sum of simple modules. If RadM is not an essensial submodule of M, then there is a non-zero submodule K of M such that $K \cap \text{Rad}M = 0$. Also $(K + \text{Rad}M)/\text{Rad}M$ is a non-zero submodule of M, so there is a simple submodule in $(K + \text{Rad}M)/\text{Rad}M \cong K$,That is, K has a simple submodule say, S with $S \cap \text{Rad}M = 0$, this implies that S is not a small submodule of M and hence there exists a proper submodule N of M such that $N + S = M$. But S is simple, thus $N \cap S = 0$ and thus $M = N \oplus S$. By $N = M/S$ is artinian. Since S is simple, then S is also artinian and hence $M = N \oplus S$ artinian.

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