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# On Primarily Hollow Module

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**Abstract**

An $R$ −module $M$ is called primarily hollow module if every primary submodule of $M$ is a small submodule in$M$. This definition extends several notions in the literature of hollow module. Some certain equivalent conditions to primarily hollow modules. Also, we verified that a primarily hollow module which contains a maximal submodule is a multiplication module and every multiplication primarily hollow module is indecomposable.

## Introduction

The concept of uniform modules was used to construct a measure of dimension for modules, now called as the uniform dimension (or Goldie dimension) of a module. Uniform dimension generalizes some, properties of the notion of the dimension of a vector space.

The hollow module is the dual notion of a uniform module. A module $M$ is called hollow if, when $A$ and $B$ are submodules of $M$ such that $A + B = M$, then either $A = M$ or $B = M$. Equivalently, if every proper submodule of M is a small submodule. These modules also admit an analog of uniform dimension, called co-uniform dimension, hollow dimension or dual Goldie dimension. The study of hollow modules and co-uniform dimension is manipulated by Fleury [2]. Fleury prospected different ways of dualizing Goldie dimension. Varadarajan, Takeuchi and Reiter's interpretations of the hollow dimension could be discussed the more natural ones [16,17]. Grzeszczuk and Puczylowski gave a definition of uniform dimension for modular lattices such that the hollow dimension of a module was the uniform dimension of its dual lattice of submodules[18]. The concept of the hollow module has been generalized to the semihollow module by Rangaswamy [9]. Rangaswamy calls a module M as semihollow if every finitely generated submodule of M is small.

Let $R$ be a commutative ring with identity and $M$ a left $R$ −module. A submodule $N$ of $M$ is called small and denoted by $N \ll M$ if $N + L \neq M$ for every proper submodule $L$ of $M$ ([19],[2]). A proper submodule $N$ of $M$ is called a primary submodule, if $rm \in N$, where $r \in R$ and $m \in M$, then $m \in N$ or $r^n M \subseteq N$ for some positive integer $n$, equivalently if for each $r \in R$ the homomorphism $\phi_r: M/N \to M/N$, defined by $\phi_r\,(m + N) = rm + N$ is either monomorphism or nilpotent ( primary submodule and consequence generalization of primary submodule can be seen in [7],[14] and [15]). An $R$ −module $M$ is said to be multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$, multiplication module was investigated by many authors (see [1],[13]). By a local ring, we mean a ring which has a unique maximal ideal. Recall that, if $P$ is a maximal submodule of $M$, then $T_P\,(M) = \{m \in$

$M: (1-p)m = 0$, for some $p \in P\}$, is a submodule of $M$. Let $M$ be a non zero multiplication module, then every proper submodule contain in a maximal submodule of $M$ and [Theorm 2.5,1]. By $rad(M)$, we mean the intersection of all maximal submodules of $M$ and if $M$ has no maximal submodule, then we set $rad(M) = M$. $M$ is called a fully primary module if every proper submodule of $M$ is a primary submodule of $M$. An $R$ −module $M$ is said to be a hollow module if every submodule of $M$ is small in $M$ [2]. A submodule $N$ of $M$ is said to coclosed, denoted by $N \leq_{cc} M$, if for every submodule $K \leq M$ with $K \subseteq N$, then $N/K \ll M/K$ implies that $N = K$ [10].

Suppose that $M$ is an $R$ −module and every primary submodule of $M$ is small in $M$, then we define these classes of modules as the primarily hollow module. In this paper, we investigated some properties of the primarily hollow module which are related to hollow, semihollow, coatomic, injective, and multiplication modules. Also, we have established that primarily hollow property is a necessary and sufficient condition to hollow property for a module under a certain condition. Moreover, we have studied some properties of the ring of the homomorphism of a primarily hollow module.

The aim of this work is to construct a generalization to the hollow module and investigate some implications of this type of module for which the hollow module does not possess these implications. Also, we proved if a ring is a primarily hollow module, then every multiplication R-modules is a primarily hollow module.

## Primarily Hollow Modules

The concept of hollow modules was introduced by Fleury [2], in 1974. In 1977, Rangaswamy [9] defined semihollow modules as a generalization of hollow modules, he called An $R$ −module $M$ is the semihollow module if $M \neq 0$ and every finitely generated proper submodule of $M$ is a small submodule of $M$. In this section we introduce and investigate some properties and characterizations of a new generalization of hollow module:

**Definition 1**. An $R$ −module $M$ is called a primarily hollow if every primary submodule of $M$ is small in $M$.

It is clear that every hollow module is a primarily hollow module, but the converse is not true in general. It is known that the $Z$ −module $Q$ is not hollow [3]. On the other hand, $Q$ contains only one prime (primary) submodule which is $\{0\}$ and clearly, $\{0\} \ll Q$, so that $Q$ is a primarily hollow module which is not hollow. In view of this example, we can say that primarily hollow modules are generalizations of hollow modules and it is obvious that, primarily hollow modules which are fully primary are hollow.

Now we give some conditions that make a primarily hollow module as a hollow module.

**Proposition 2.** Let $M$ be a primarily hollow module. If $M$ is coatomic, then it is a hollow module.

**Proof.** Let $N$ be a proper submodule of $M$ and as $M$ is coatomic, $N$ is contained in a maximal submodule say $L$. Since every maximal submodule is a prime submodule and hence primary, so $L \ll M$ thus $N \ll M$ **[11]**. Hence, $M$ is hollow.

**Proposition 3**. Every finitely generated primarily hollow module is hollow.

**Proof.** Suppose that $M$ is a finitely generated primarily hollow module. since $M$ is a finitely generated module then every proper submodule $L$ contains in a maximal submodule$N$and every maximal is primary, so $N \ll M$ hence $L \ll M$ and hence $M$ is hollow.

We prove that primarily hollow modules are preserved under epimorphisms.

**Proposition 4**. Let $M_1$ and $M_2$ be $R$ −modules and $f: M_1 \to M_2$ be an epimorphism. If $M_1$ is primarily hollow, then so is $M_2$.

**Proof**. Clearly, $f(M_1) = M_2$. Let $N$ be a primary submodule of $M_2$, we must show that $N \ll M_2$. Clearly we have, $f^{-1}(N) \leq M_1$, then , we have $f^{-1}(N)$ is a primary submodule of $M_1$ and as $M_1$ is primarily hollow, we get $f^{-1}(N) \ll M_1$, which gives that $f(f^{-1}(N)) \ll M_2$**[11]** . Thus, $M_2$ is also primary hollow.

**Corollary 5**. If $M$ is a primarily hollow module and $N$ is a submodule of $M$, then $\frac{M}{N}$ is also a primarily hollow module.

**Proof.** Since $\pi: M \to \frac{M}{N}$ is an epimorphism, so the result follows directly from **Proposition 4.**

**Corollary 6**. A direct summand of a primarily hollow module is also primarily hollow.

**Proof.** Let $M$ be a primarily hollow module with $M = A \oplus B$, where $A, B \leq M$. Clearly we have, the projections $P_A: M = A \oplus B \to A$ and $P_B: M = A \oplus B \to B$ are epimorphisms, so by **Proposition 4**, we get each of $A$ and $B$ is primarily hollow and this completes the proof.

Now, we give an example of a submodule of a primarily hollow module which is not primarily hollow. It is obvious that, the $Z$ −module $Q$ is primarily hollow. It is clear that $Z$ is a submodule of $Q$ which is not primarily hollow since we have $< 2 >$ is a prime submodule (and hence a primary submodule) of $Z$ with $< 2 > + < 3 > = Z$ while $< 3 > \neq Z$, that means $Z$ is not primarily hollow.

**Proposition 7**. Let $M$ be an $R-$ module and let $K$ be a coclosed submodule of $M$. If every primary submodule of $K$ is a small submodule of $M$, then $K$ is a primarily hollow module.

**Proof.** We must show that $K$ is a primarily hollow module. Let $X$ be a primary submodule of $K$, so that $X \ll M$. To show $X \ll K$. Let $X + A = K$ and $\frac{N}{A}$ be any submodule of $\frac{M}{A}$ such that $\frac{K}{A} + \frac{N}{A} = \frac{M}{A}$. Now we have $\frac{(K+N)}{A} = \frac{M}{A}$, from which we get $M = K + N = X + A + N$, this gives $X + N = M$ and as $X \ll M$, we get $N = M$, so that $\frac{N}{A} = \frac{M}{A}$, hence $\frac{K}{A} \ll \frac{M}{A}$. As $K$ is coclosed, we get $K = A$, so that $X \ll K$. Hence, $K$ is a primarily hollow module.

**Theorem 8.** A primarily hollow module which contains a maximal submodule is a multiplication module.

**Proof.** Suppose that $M$ is a primarily hollow module and $N$ is a maximal submodule of $M$, then $N$ is prime and hence a primary submodule of $M$ and as $M$ is primarily hollow, we get $N \ll M$. Now, as $N \neq M$, there exists $x \in M \backslash N$. Now, $Rx$ is a submodule of $M$ and $Rx \nsubseteq N$, then $Rx + N = M$, this gives $Rx = M$. If $P$ is a maximal ideal of $R$, then by **[1, Theorem 1.2]**, we get $(1-q)M \subseteq Rx$ for some $q \in P$, so that $M$ is $P$ −cyclic and that $M$ is a multiplication module.

**Theorem 9.** Let $M$ be a multiplication module. If $M$ is primarily hollow, then $M$ is cyclic.

**Proof.** Since, $M$ is a multiplication module, so by **[1,Theorem 5]**, it has a maximal submodule say $L$, then there exists $x \in M \backslash L$. Now, for the submodule $Rx$ of $M$, if $Rx \subseteq L$, then we get $x \in L$, which is a contradiction, so we must have $Rx \nsubseteq L$, this gives $Rx + L = M$ and as $L \ll M$, we get $Rx = M$. Hence $M$ is cyclic.

**Proposition 10.** Let $M$ be a multiplication $R$ −module, then $M$ is a hollow module if and only if $M$ is a primarily hollow module.

**Proof.** The proof of the necessity part is obvious and for the sufficiency part, we suppose that, $M$ is a primarily hollow module. Let $N$ be a proper submodule of $M$. Since, $M$ is multiplication, so there exists a maximal submodule $L$ of $M$ such that $N \subseteq L \subset M$ and since every maximal submodule is primary, we have $L \ll M$, so by **[11]**, we have $N \ll M$. Hence $M$ is a hollow module.

**Proposition 11.** Let $R$ be a commutative ring with identity and $M$ be a faithful multiplication module, then $K$ is a primary submodule of $M$ if and only if there exist a primary ideal $I$ of $R$ such that $K = IM$ and $M \neq IM$.

**Proof.** (⇐) Let $I$ be a primary ideal of $R$ such that $K = IM$ and $M \neq IM$, then by **[6, Corollary 1]**, we get $K$ is primary.

(⇒) Let $K$ be a primary submodule of $M$, then by **[7]**, we have $(K: M)$ is a primary ideal of $R$ and $K = (K: M)M$ and clearly $M \neq (K: M)M$.

In the following result we show that if $R$ is a primarily hollow ring, then the class of multiplication modules is a subclass of primarily hollow modules.

**Proposition 1** Let $M$ be a faithful multiplication $R$ −module, then $M$ is a primarily hollow module if and only if every primary ideal of $R$ is small.

**Proof.** (⇒) Let $M$ be a primarily hollow module and $I$ be a primary ideal of $R$, then by **Proposition 11**, we have $IM$ is primary submodule of $M$, so that $IM \ll M$. Let $J$ be any ideal of $R$ such that $I + J = R$, then $IM + JM = (I + J)M = RM = M$, then we have $JM = M$. If $J \neq R$, then $J \subseteq P$ for some maximal ideal $P$ of $R$, this gives $M = PM$ and by **[6, Theorem 5]**, we have $PM$ is a maximal submodule of $M$, which is a contradiction. Hence $J = R$ and that $R$ is a hollow $R$ −module.

(⇐) Suppose that every primary ideal of $R$ is small. Let $N$ be a primary submodule of $M$ such that $N + L = M$. As, $N$ is primary, by **[7]**, we have $(N: M)$ is a primary ideal of $R$, so it is small in $R$. Now, $M = N + L = (N: M)M + (L: M)M = \big((N: M) + (L: M)\big)M$, then we get $(N: M) + (L: M) = R$, this gives $(L: M) = R$, so that we get $L = (L: M)M = RM = M$. Thus, we get $N \ll M$ and that $M$ is a primarily hollow module.

**Proposition 13.** Let $M$ be Noetherian and a semihollow module, then $M$ is a primarily hollow module.

**Proof**. Let $N$ be a primary submodule of $M$. As $M$ is Noetherian, $N$ is finitely generated, so we get $N \ll M$. Hence $M$ is a primarily hollow module.

**Proposition 14.** Let $M$ be a multiplication module. Then $M$ is local if and only $M$ is primarily hollow and $rad(M)$ is a maximal submodule of $M$.

**Proof.** Suppose that $M$ is a local module, then by **[3]**, we get $M$ is hollow and has a unique maximal submodule $K$, so that $rad(M) = K$, that means $rad(M)$ is a maximal submodule of $M$**.** Conversely, suppose that $M$ is primarily hollow and $rad(M)$ is a maximal submodule of $M$. As $M$ is a multiplication module, we get $M$ is hollow which by **[3, Lemma1]** implies that $rad(M)$ is a unique maximal submodule. Hence $M$ is a local module

Next, we prove that the locality property of a ring is a necessary and sufficient condition for primarily hollow modules to be faithful multiplication modules.

**Proposition 15.** Suppose that $R$ is a commutative ring with identity, then $R$ is local ring if and only if every faithful multiplication $R$ −module is primarily hollow module.

**Proof.** (⇒) Suppose that $R$ is a local ring with the unique maximal ideal $P$. Let $M$ be a faithful multiplication $R$ −module and $N$ be a primary submodule of $M$, then there exists a maximal submodule $K$ of $M$ such that $N \subseteq K$, then by**[1,Theorem 5]**, we get $K = PM \neq M$ for some maximal ideal $P$ of $R$. If $L$ is a submodule of $M$ such that $K + L = M$ and $L = IM$ where $I$ is an ideal of $R$. If $I \neq R$, then $I \subseteq P$ and $PM = (P + I)M = PM + IM = K + L = M$, which is a contradiction. So $I = R$ and then $L = IM = RM = M$, thus $K$ is small in $M$, this gives that $N$ is also a small submodule of $M$. Hence $M$ is primarily hollow.

(⇐) Let $M$ be a faithful multiplication primarily hollow module. To show $R$ is a local ring. If $P_1$ and $P_2$ are any two distinct maximal ideals of $R$, then we have $P_1 + P_2 = R$, from which we get $P_1M + P_2M = (P_1 + P_2)M = RM = M$. By **[1,Theorem 5]**, we have $P_1M$ and $P_2M$ are maximal submodules of $M$, so they are primary and since $M$ is primarily hollow, so $P_1M$ and $P_2M$ are small in $M$, then $P_1M + P_2M = M$ implies that $P_1M = M$ or $P_2M = M$ which is contradiction. Hence $R$ must have a unique maximal ideal.

From Proposition 15, we obtain the following result:

**Corollary 16.** For a faithful multiplication $R$ −module $M$ the following statements are equivalent:

**(1)** $R$ is a local ring.

**(2)** $R$ is a primarily hollow ring.

**(3)** $M$ is a primarily hollow module.

**Proposition 17.** Let $M$ be a primarily hollow module. If for every proper submodule $N$ of $M$ and for each $r \in R$ the homomorphism $\phi_r: \frac{M}{N} \to \frac{M}{N}$, defined by $\phi_r(m+N) = rm + N$ is either a monomorphism or nilpotent, then $M$ is a hollow module.

**Proof.** Let $N$ be a proper submodule of $M$ and as $M$ is a primary hollow module, by **[7]**, we get $N$ is a primary submodule of $M$, so that $N \ll M$. Hence $M$ is a hollow module.

In the result proposition we prove that if the external direct summand of two modules is primarily hollow, then both the modules must be primarily hollow modules.

**Proposition 18.** Let $M = M_1 \oplus M_2$, where $M, M_1, M_2$ are $R$ −modules. If $M$ is primarily hollow, then $M_1$ and $M_2$ are also primarily hollow.

**Proof**: Define an $R$ −morphism $P_i: M \to M_i$ by $P_i(m) = m_i$, where $m = m_1 + m_2$ for $m_i \in M_i$ and $i = 1$, We can easily show that $P_i$ is an epimorphism, then by **Corollary 3**, we have $M_1$ and $M_2$ are primarily hollow modules.

The converse of **Proposition 18** is not true in general. For example consider the module $Z_p \oplus Q$, where $Z_p$ is the module of integers modulo $p$, where $p$ is a prime integer and $Q$ is the module of rational numbers. Then , $Z_p$ and $Q$ are primary hollow modules. Now we have, $\{0\} \oplus Q$ is a primary submodule of $Z_p \oplus Q$ and $\{0\} \oplus Q + Z_p \oplus \{0\} = Z_p \oplus Q$ but $Z_p \oplus \{0\} \neq Z_p + Q$, so that $\{0\} \oplus Q$ is not small in $Z_p \oplus Q$. Hence $Z_p \oplus Q$ is not primarily hollow, while neither $Z_p$ is primarily hollow nor $Q$.

It is easy to show that if $M$ is an injective module and $M_1$ and $M_2$ are primarily hollow, then $M$ is also a primary hollow module.

**Proposition 19.** If $M$ is an $R$ −module, then $M$ is a primarily hollow module and finitely generated if and only if $M$ is cyclic and has a unique maximal submodule.

**Proof.** ($\Rightarrow$) Let $M$ be a primarily hollow module and finitely generated, then there exist $x_1, x_2, \ldots, x_n \in M$ such that $M = Rx_1 + \cdots + Rx_n$, since every finitely generated module is a multiplication module, then $M$ is multiplication. If $M \neq Rx_1$, then there exists a primary submodule $N_1$ of $M$ such that $Rx_1 \subseteq N_1$ so $M = N_1 + Rx_2 + \cdots + Rx_n$ and since $M$ is primarily hollow, then $N_1$ is small in $M$. Thus $M = Rx_2 + \cdots + Rx_n$. So on we delete the summands one by one until we get $M = Rx_i$ for some $1 \leq i \leq n$. Hence $M$ is cyclic. Next, suppose that $M$ has two maximal submodules say $M_1$ and $M_2$, then $M = M_1 + M_2$ and since $M$ is primarily hollow, then $M = M_1$ which is a contradiction. Hence, $M$ has a unique maximal submodule.

($\Leftarrow$) Suppose that $M$ is cyclic and has a unique maximal submodule $P$, then $M$ is finitely generated and to show $M$ is primarily hollow, let $N$ be a primary submodule of $M$, then $N \subseteq P$. If $L$ is a submodule such that $N + L = M$. If $L \neq M$, then $L \subseteq P$, so that we get $M = N + L \subseteq P$, which gives $P = M$, that is a contradiction and so that $L = M$. Hence, $N$ is a small submodule in $M$ and that $M$ is primarily hollow.

**Proposition 20.** If $M$ is a primarily hollow module and $N$ is a proper submodule of $M$ such that $\frac{M}{N}$ is finitely generated, then $M$ is a finitely generated hollow module.

**Proof.** Since $\frac{M}{N}$ is finitely generated, then there exist $x_1, x_2, \ldots, x_n \in M$, such that $\frac{M}{N} = Rx_1 + \cdots + Rx_n + N$. We must show that $M$ is finitely generated so let $x \in M$, then $x + N \in \frac{M}{N} = Rx_1 + \cdots + Rx_n + N$, so there exist $r_1, \ldots, r_n \in R$ such that $x + N = r_1x_1 + \cdots + r_nx_n + N$, then we have $x - r_1x_1 + \cdots + r_nx_n = n$ for some $n \in N$, then $x = r_1x_1 + \cdots + r_nx_n + n \in Rx_1 + \cdots + Rx_n + N$ so $M = Rx_1 + \cdots + Rx_n + N$ and since $\frac{M}{N}$ is finitely generated, then it has a maximal submodule (and hence a primary submodule) say $\bar{P}$, then there exists a primary submodule $P$ of $M$ such that $N \subseteq P$ and $\bar{P} = \frac{P}{N}$, so we have $M = Rx_1 + \cdots + Rx_n + N = Rx_1 + \cdots + Rx_n + P$ and since $P$ is primary and $M$ is primarily hollow, we have $M = Rx_1 + \cdots + Rx_n$.

Hence, $M$ is finitely generated. Since a finitely generated primarily hollow module is a hollow module, so $M$ is hollow.

We know from **[3]** that, every hollow module is indecomposable and now we prove that multiplication primarily hollow modules are indecomposable.

**Proposition 21.** Every multiplication primarily hollow module is indecomposable.

**Proof.** Let $M$ be a multiplication primary hollow module. Suppose that $M$ is decomposable, so there exist proper submodules $K$ and $L$ of $M$ such that $M = K \oplus L$. Since $M$ is multiplication, then by **[1, Theorem 5 ]**, there exist maximal submodules $A$ and $B$ of $M$ for which $L \subseteq A$ and $K \subseteq B$**,** then we have $M = A + B$ and as $M$ is primarily hollow, we get $A = M$ or $B = M$, which is a contradiction. Hence $M$ is indecomposable.

**Proposition 2** Let $M$ be a multiplication module, then $M$ is a primarily hollow module if and only if every non-zero factor module of $M$ is indecomposable.

**Proof.** (⇒) Let $M$ be a primary hollow module and $\frac{M}{N}$ a nonzero factor module of $M$, then by **[Corollary 5]**, we have $\frac{M}{N}$ is primarily hollow and multiplication and by **Proposition 21**, we get $\frac{M}{N}$ is indecomposable.

(⇐) Suppose that every nonzero factor module of $M$ is indecomposable, then $M$ is hollow and hence a primarily hollow module.

**Proposition 23.** Let $M$ be an $R$ −module and $I$ a proper ideal of $R$ such that $\cong \frac{R}{I}$ , then $M$ is a primarily hollow module if and only if $I$ is contained in a unique maximal ideal.

**Proof**. Let $M$ be primarily hollow and suppose that $I$ is contained in two maximal (primary) ideals, say $I_1$ and $I_2$, then $I_1 + I_2 = R$, then $\frac{I_1}{I} + \frac{I_2}{I} = \frac{R}{I}$. Since, $M \cong \frac{R}{I}$, so $\frac{R}{I}$ is a primarily hollow module and as $\frac{I_1}{I}$ and $\frac{I_2}{I}$ are primary, they are small in $\frac{R}{I}$, so we get $\frac{I_1}{I} = \frac{R}{I}$ and this gives $I_1 = R$, which is a contradiction. Hence $I$ must contained in a unique maximal ideal. Conversely, suppose that $I$ is contained in a unique maximal ideal $P$ of $R$, then by **[2]** $\frac{R}{I}$ is hollow and hence primarily hollow .

**Corollary 24.** Let $\frac{R}{I}$ be a primarily hollow module and $I$ is an ideal of $R$, then $End(\frac{R}{I})$ is a local ring.

**Proof.** It is clear that $End\left(\frac{R}{I}\right) \cong \frac{R}{I}$ and by **Proposition 23**, we have $\frac{R}{I}$ has a unique maximal ideal $\frac{M}{I}$. Hence $End\left(\frac{R}{I}\right)$ has a unique maximal ideal and thus it is local.

**Proposition 25.** Let $I$ be a proper ideal of $R$ such that $\frac{R}{I}$ is a multiplication primarily hollow module. If $\frac{R}{I}$ is either projective or injective, then $End(\frac{R}{I})$ is local.

**Proof.** Since we have $\frac{R}{I}$ is a multiplication primarily hollow module then by**[1,Theorem 5]** has auniqe maximal submodule and hence $\frac{R}{I}$ is local.Now, by **[12, Theorem 4]** $End(\frac{R}{I})$ is local. a For the second part, since $\frac{R}{I}$ is multiplication primarily hollow, then by **Proposition 22**, $M$ is indecomposable and it is well-known that an indecomposable injective has a local ring of endomorphisms.

In the following result, we investigate the locality of the ring $End(M)$ where $M$ is a primary hollow module.

**Proposition 26.** If $M$ is a primarily hollow module with the property that every epimorphism on $M$ is an isomorphism and whenever $f(M) \neq M$, then $f(M)$ is contained in a primary submodule of $M$ and $End(M)$ is local.

**Proof.** Let $S$ be the set of all non-units of $End(M)$, we have to show $S$ is a maximal ideal of $End(M)$. If $S = \phi$, then $End(M)$ is a simple ring which is local, then we suppose $S \neq \phi$. If $f, g \in S$, then $f$ and $g$ are not unit. If $f+g$ is unit, then we get $(f+g)(M)=M$, then $f(M)+g(M)=M$ since $f(M)\neq M$ and $g(M)\neq M$, then there exist primary submodules $N_1, N_2$ of $M$ such that $f(M)\subseteq N_1$ and $g(M)\subseteq N_2$. So we get $N_1+N_2=M$, then $N_1=M$ or $N_2=M$ which is a contradiction. Then we have $f+g\in S$. If $k\in End(M)$ and $f\in S$ to show $k\circ f\in S$. If $k\circ f\notin S$, then $k\circ f$ is a unit that is $k\circ f$ is onto and one to one, then $f$ is onto thus by our assumption $f$ is unit which is contradiction, then $k\circ f\in S$. Hence $S$ is ideal of $End(M)$. And to show $S$ is a maximal ideal suppose $S\subseteq L\subseteq End(M)$ and $S\neq L$, then there exist $f\in L$ and $f\notin S$ that means $L$ contains a unit of $End(M)$, then $End(M)=L$. If $D$ is another maximal ideal of $End(M)$, then there exist $g\in D$ and $g\notin S$ that means $D$ contains a unit of $End(M)$, then $End(M)=D$ which is a contradiction that is $End(M)$ has a unique maximal ideal . Hence $End(M)$ is local.

If $R$ is a ring then the idealizer of a left ideal $I$ is the set $\rho(I)=\{a\in R; Ia\subseteq I\}$.

**Corollary 27.** Let $R$ be a ring with identity and $M$ be an $R-$module which is primarily hollow and $M\cong\frac{R}{I}$ for a proper ideal $I$of $R$. Suppose $R$ satisfies the following chain condition. For every $a\in\rho(I)$, the sequence of proper ideals $I\subseteq(I:a)\subseteq(I:a^2)\subseteq\cdots$ eventually stabilizes and whenever $f\left(\frac{R}{I}\right)\neq\frac{R}{I}$ implies that $f\left(\frac{R}{I}\right)$ contained in a primary submodule of $\frac{R}{I}$ for any $f\in End\left(\frac{R}{I}\right)$, then $End(M)$ is local.

**Proof.** The proof follows directly by **Proposition 26**, and **[2].**

Let Xbe an $R-$module and $Y_1, Y_2$ be submodules of $X$. We say that $Y1$ and $Y2$ are correlated if $Y1+H=X$ if and only if $Y2+H=X$ for some submodule, $H$ of $X$.

**Theorem 28**. Let $M$ be an $R-$module such that $M\cong\frac{R}{I}$, where $I$ is a left ideal of $R$ and $P$ be a maximal ideal such that $I\subseteq P$. Then $M$ is a primarily hollow module if and only if $Rx+I=R$, for each $x\notin P$.

**Proof.** (⇒) Let $M$ be primarily hollow module, that is $\frac{R}{I}$ is primary hollow, then by **Proposition 23**, $I$ contained in a unique maximal ideal $P$ of $R$, then there exists $x\notin P$ so $Rx\nsubseteq P$, then $Rx+P=R$, then $\frac{R}{I}=\frac{Rx+P}{I}=\frac{Rx+I}{I}+\frac{P}{I}$. But $\frac{P}{I}$ is a maximal ideal of $\frac{R}{I}$ and $\frac{R}{I}$ is primary hollow, then we get $\frac{R}{I}=\frac{Rx+I}{I}$, then $Rx+I=R$ **[2]**.

(⇐) Let $Rx+I=R$, for each $x\notin P$, then $Rx+P=R$. If $I\subseteq P'$, where $P'$ is another maximal ideal of $R$, then $P+P'=R$, since $I$ is correlated to $P$, then by **[2]**, we have $I+P'=R$, then $P'=R$ which is a contradiction. Thus $I$ is contained in a unique maximal ideal, then by **Proposition 23**, we get $M$ is a primarily hollow module.

**Theorem 29.** If $M$ is primarily hollow and finitely generated, then $M$ is local.

**Proof**. If $N, K$ are two maximal submodules of $M$, then $N+K=M$, since $M$ is primarily hollow, then $N=M$ or $K=M$ which is a contradiction. Hence $M$ is local.

**Theorem 30.** If $M$ is a module and every primary submodule of $M$ has a small coessential submodule, then $M$ is primarily hollow.

**Proof.** Let $P$ be a primary submodule of $M$, then there exist a small submodule $K$ of $M$ such that $\frac{P}{K}\ll\frac{M}{K}$. If $P+L=M$, then $\frac{P+L}{K}=\frac{M}{K}$, then $\frac{P}{K}+\frac{K+L}{K}=\frac{M}{K}$, then $\frac{K+L}{K}=\frac{M}{K}$, that means $K+L=M$ so $L=M$. Hence $M$ is a primary hollow module.

**Theorem 31.** If $M$ is an injective $R-$module, then $M$ is primarily hollow if and only if every direct summand of $M$ is a primary hollow submodule.

**Proof.** Let $M$ be primarily hollow module and $A$ be a direct summand of $M$, then there exists a submodule $B$ of $M$ such that $M=A\oplus B$, and since $M$ is injective, let $P_r: M\to A$, defined by $P_r(x)=a$, for $x=a+b$, for $a\in A, b\in B$ and $I: M\to M$, is the identity homomorphism on $M$, then we can easily show that $I, P_r$ are onto, since $M$ is injective, then there exist $g: A\to M$ such that $gP_r=I$ and since $I$ and $P_r$ are onto, then $g$ is also onto. If $M$ is primarily hollow, then $\frac{M}{B}$ is primarily hollow by **proposition 4**, then $\frac{M}{B} \cong A$ hence $A$ is primarily hollow .

**Theorem 3** If $M$ is an injective $R$ −module, then $M$ is primarily hollow if and only if $\frac{M}{A}$ is primarily hollow module for every submodule $A$ of $M$.

**Proof.** (⇒) Clear by **proposition 4.**

(⇐)Let $M$ be injective, $\frac{M}{A}$ be primarily hollow and $\pi: M \to \frac{M}{A}$ is natural mapping and $I: M \to M$ be the identity mapping, then there exist $g: \frac{M}{A} \to M$ such that $g \circ \pi = I$, then $g\left(\frac{M}{A}\right) = M$, then $M$ is primarily.

**Theorem 33.** If $M$ is an injective $R$ −module and $g: M \to N$ is an epimorphism, then $M$ is primarily hollow if and only if the $N$ is a primarily hollow module.

**Proof.** (⇒) Clear. by **proposition 4**.

(⇐)Let $M$ be injective, $N$ be primarily hollow and $g: M \to N$ is an epimorphism and $I: M \to M$ be the identity mapping, then there exist $f: N \to M$ such that $f \circ g = I$, then $f(N) = M$, then $M$ is primarily.

If $R = R_1 \times R_2$,where $R_i$ is a commutative ring with identity and $M_i$ is an $R_i -$ module, then $M = M_1 \times M_2$ forms an $R -$ module with the operation $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$, for $r_i \in R_i$ and $m_i \in M_i$.

**Lemma 34.** If $R, M$ are defined as above, then we have the following

(1) If $N_i$ is a primary submodule of $M_i$ for $i = 1,2$, then $N_1 \times M_2$ and $M_1 \times N_2$ are primary submodule of $M$.

(2) If $N_1 \times N_2$ is a primary submodule of $M$, then $N_i$ is a primary submodule of $M_i$,for $i = 1,$

**Proof.**

1- Let $N_1$ be primary in $M_1$and $(r_1, r_2)(m_1, m_2) \in N_1 \times M_2$, then $(r_1 m_1, r_2 m_2) \in N_1 \times M_2$, than $r_1 m_1 \in N_1$. Since $N_1$ is primary submodule of $M_1$, then $m_1 \in N_1$ or $r_1^n M_1 \subseteq N_1$. If $m_1 \in N_1$, then $(m_1, m_2) \in N_1 \times M_2$ otherwise we get $r_1^n M_1 \subseteq N_1$ and always we have $r_2^n M_2 \subseteq M_1$, then we get $(r_1^n, r_2^n) M \subseteq N_1 \times M_2$. Hence $N_1 \times M_2$ is a primary submodule of $M$. By the same way we can show that $M_1 \times N_2$is primary submodule of $M$.

2- Suppose $N_1 \times N_2$ and $r_1 m_1 \in N_1$and $r_2 m_2 \in N_2$, then $(r_1, r_2)(m_1, m_2) \in N_1 \times N_2$, then $(m_1, m_2) \in N_1 \times N_2$ or $(r_1^n, r_2^n) M \in N_1 \times N_2$. Hence we get $m_1 \in N_1$or $r_i^n M_i \subseteq M_i$. Thus $N_i$ is primary submoduleof $M_i$.

**Proposition 35.** Suppose that $M_i$ is an $R_i$ −module if $M_1 \times M_2$ is primarily hollow module then $M_i$ is primarily hollow module for $i = 1,$

**Proof.** Let $N_1$ be a primary submodule of $M_1$, then by **Lemma 34**, we get $N_1 \times M_2$ is primary submodule of $M_1 \times M_2$, then $N_1 \times M_2$ is a small submodule of $M_1 \times M_2$. Now let $L_1$ be submodules of $M_1$ such that $N_1 + L_1 = M_1$, then $M_1 \times M_2 = (N_1 + L_1) \times M_2 = (N_1 \times M_2) + (L_1 \times M_2)$, then we get $M_1 \times M_2 = (L_1 \times M_2)$, then $M_1 = L_1$. Hence $M_1$ is primarily hollow module. By the same way we can show that $M_2$ is primarily hollow module.

The converse of **Proposition 35**, is not true in general, for example consider $Z_2$ as a $Z$ −module, then it is clear $Z_2$ is a primarily hollow module. But $Z_2 \times Z_2$ is not a primarily hollow $Z \times Z$ −module.

According to **[6]**, if $M$ is an $R$ −module, then $M$ is called a $Q$ −module if every proper submodule $N$ of $M$ either is primary or has a primary factorization $N = Q_1 Q_2 \ldots Q_n N^*$, where $Q_1, Q_2, \ldots, Q_n$ are primary ideals of $R$ and $N^*$ is a primary submodule in $M$.

From Proposition 36, we get the following result:

**Proposition 36.** Let $M$ be an $R$ −module. If $M$ is a $Q$ −module, then $M$ is a hollow module if and only if $M$ is a primarily hollow module.

**Proof**. (⇒) Is clear.

(⇐) Let $M$ be primarily hollow, to show $M$ is hollow. Let $N$ be a proper submodule of $M$, then we have $N$ is primary or $N = Q_1Q_2 \dots Q_nN^*$, where $Q_1, Q_2, \dots, Q_n$ are primary ideals of $R$ and $N^*$ is a primary submodule in $M$. If $N$ is primary, then $N \ll M$ and if $N = Q_1Q_2 \dots Q_nN^*$, where $N^*$ is a primary submodule in $M$, then $N^* \ll M$. If $L$ is a submodule of $M$ such that $N + L = M$ and since $N = Q_1Q_2 \dots Q_nN^*$, so $N \subseteq N^*$, then $N^* + L = M$, then $L = M$. So we have $N \ll M$. Hence, $M$ is hollow.

**Corollary 37**. If $R$ is a $Q$ −ring, then every faithful multiplication $R$ −module $M$ is primary hollow if and only if $M$ is hollow.

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